A Robust Reference-Dependent Model for Speculative Bubbles*

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Abstract

We present a robust model of speculative bubbles by introducing loss-averse reference-dependent preferences by Koszegi and Rabin (2006) into the framework of Allen, Morris and Postlewaite (1993), where in equilibrium, asymmetrically-informed rational investors buy overvalued assets, hoping to sell them to less informed agents before the crash occurs. With reference-dependent preferences, the asset price may not necessarily be observable to agents when there is no trade. However, this is never the case with classical preferences, as shown in the paper. Incorporating the classical model as a special case, we generalize the notion of bubbles to allow for the analysis in the case of a silent market with unobservable prices, and our model is able to generate strong bubbles robust to moderate perturbations in parameters without the need for stronger conditions as suggested in previous literature. Assuming for simplicity that dividends can only take on two values, we construct an example of a robust reference-dependent bubble which is not robust in the classical setting, and we also show that the positive results regarding the limit of the bubble size and bubble frequency in the classical setting are preserved in our framework. Our main results and economic implications remain valid in more general settings.

Keywords: Speculative Bubbles, Robustness, Rational Expectations Equilibrium, Reference Dependence

JEL Classifications: D81, D83, D84, G02, G12

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Normally sensible people drift into behavior akin to that of Cinderella at the ball. They know that overstaying the festivals will eventually bring on pumpkins and mice... participants all plan to leave just seconds before midnight... (But) they are dancing in a room in which the clocks have no hands.

—Warren Buffett

1 Introduction

The last two decades have witnessed at least two dramatic boom-and-bust episodes – the dot-com bubble (Ofek and Richardson, 2003) and the subprime crisis (Varadarajan, Christiano and Keho, 2008), which seem like replications of the stories in Kindleberger and Aliber (2011), including the Dutch tulip mania (1634-1637), the Mississippi bubble (1719-1720) and the South Sea bubble (1720).[1] Similar phenomena have also been observed in the laboratory environment (Dufwenberg, Lindqvist and Moore, 2005; Moinas and Poufet, 2012; Lugovskyy et al,2014; Bao, Hommes and Makarewicz, 2017, among others) where bubbles occur with asymmetrically informed agents aware of the possibilities of both riding the bubble and getting stuck, or bubbles are robustly generated in markets with positive expectation feedback.

Despite its nearly unambiguous existence and prevalence in empirical studies, the phenomena of bubbles seem difficult to explain using classical economic theory. There is a large strand of literature trying to introduce the ideas of overlapping generations to rationalize bubbles (Tirole, 1985; Fahri and Tirole, 2012; Martin and Ventura, 2010). We refer to this type of bubble as an "investment bubble" in the sense that the asset serves as a store of value and may grow slowly without bursting, or alternatively may burst because of the insufficiency of cash (Caginalp and Ilieva, 2008). This can be regarded as a type of moderate-scale bubbles from the long-term perspective. However, the bubbles mentioned in the beginning of the paper typically involve an intense crash, calling for a distinct definition of bubbles from the short-term perspective. Following Conlon (2004) and Doblas-Madrid (2012), we characterize this type of bubble as a "speculative bubble" where rational agents consciously buy the over-valued assets in the hope of selling them to a greater fool before the assets crash. In this paper we narrow our focus to speculative bubbles.

Tirole (1982) has shown that with a homogeneous setup, rational expectations equilibrium is incompatible with speculative bubbles. In this sense, it is necessary to introduce some form of heterogeneity in order to generate bubbles, the idea of which is aptly captured by the opening

[1] Other examples may include the 2005-2007 and 2008-2009 Chinese stock market bubbles (Jiang et al, 2010). However, there is still some controversy about whether these can be classified as strong bubbles.

quotation by Warren Buffet: Investors hold the over-priced asset in the expectation of getting a higher payoff by selling it to a "greater fool" and quitting the market just before the bubble bursts, but at the same time it is possible that they may stay too long to actually successfully sell the asset. Allen, Morris and Postlewaite (1993, henceforth referred to as AMP) precisely captured this intuition in their finite-horizon bubble model with asymmetric information and short sale constraint. By their notion of "strong bubbles", every trader knows that the asset is over-priced with certainty, however, they still would like to hold the asset because there is uncertainty about other traders' knowledge of this over-pricing phenomenon. The AMP framework has been well adopted in the literature on rational bubbles, given its success in explaining the existence of bubbles from the perspective of information economics (Conlon, 2004; Zheng, 2014; Conlon 2015; Lien, Zhang and Zheng, 2015; among others). However, it has also been shown that the bubble equilibria in AMP model are fragile and not very robust to small perturbations in payoff or belief parameters (Zheng, 2014; Conlon and Zheng, 2013). Intuitively, to support rational bubbles, public signals (prices) should not reveal too much information; that is, certain states of the world need to be indistinguishable from one another in observing the market price. Also, with risk neutrality and competitive markets, players should be indifferent between selling or buying an additional unit of the asset in equilibrium. This necessary condition translates into a system of equalities for parameters under the classical AMP setup, and thus fails to hold when there are small asymmetric perturbations in the values of parameters such as priors or dividends, since the players may find it strictly better off to trade and force the equilibrium prices to vary in previously indistinguishable states, which in turn ruins the proposed information structure that supports the rational bubble.

In order to take into account the main concern of the bubbles' robustness issue, we extend the AMP framework to allow for a more general type of utility – reference-dependent loss-averse utility in this paper, and show that the bubbles are no longer fragile when agents have such preferences.[2] The ideas of reference dependence was first observed and formulated in the Kahneman and Tversky's seminal paper on prospect theory (1976) and has been studied in various fields (for example, Ericson and Fuster, 2011; Eil and Lien, 2014; Humphreys and Zhou, 2015; Lien and Zheng, 2015, among many others). Koszegi and Rabin (2006, 2007) study the loss aversion feature of reference-dependent preferences by introducing an extra gain-loss utility term into the traditional consumption utility function and set a consumer's recent rational expectations about outcomes as her reference point. As for empirical justifications of using expectations as the reference point,

[2]Since the classical reference-independent preferences are only a special case in the class of reference-dependent preferences, the nonbusiness issue for bubbles in AMP framework will no longer be too much of a concern, as long as it can be shown that with reference-dependent preferences bubbles are in general robust. This approach of extending an existing classical model to incorporate realistic behavioral features, to provide new insights and different results under different scenarios, is named as "Portable Extensions of Existing Models" ("PEEMs") by behavioral economist Matthew Rabin (Rabin, 2013).

it has been well observed that expectations influence the trading behavior in general (for example, List, 2003; Ericson and Fuster, 2011) and the bubble formation in particular (Hommes et al, 2008; Husler, Sornette and Hommes, 2013; Bao, Hommes and Makarewicz, 2017, among others) in the lab environment.[3] Convinced by the empirical and experimental evidence, we follow Koszegi and Rabin (2006, 2007), adopt the loss aversion type of reference-dependent preferences, and assume rational expectations as the reference point for every trader in our model. Henceforth, for convenience, we refer to such a behaviorial approach to modeling preferences as the KR approach and the relevant preferences as the KR preferences.

In this paper, we present a robust model of speculative bubbles by introducing the KR preferences into the AMP framework, where in equilibrium, asymmetrically-informed rational investors buy overvalued assets, hoping to sell them to less-informed investors before the crash occurs. Incorporating the classical model as a special case, our model is able to generate speculative bubbles which are robust to moderate perturbations in parameters without a need for the stronger conditions suggested in previous literature (Zheng, 2014). Our baseline model assumes assets with binary valued dividends, and shows that the size of a robust bubble can approach the highest dividend level in a particular Rational Expectations Equilibrium (REE) and that a robust bubble can appear almost for sure in equilibrium (Lien, Zhang and Zheng, 2015). Our main results and economic implications remain valid in more general settings, as shown in the appendix.

The main contribution of this paper to the bubble literature is to resolve the equilibrium fragility problem in AMP (1993) without loss of generality, by assuming that agents' preferences are loss-averse and reference-dependent, with an additional term in the utility function which measures the sensation of gain and loss relative to some reference point. In fact, the classical AMP model serves as a special case of our model: When the agents only care about the absolute level of consumption or when there is no uncertainty, the two models naturally coincide with each other since the gain-loss utility simply acts as an additive term to the fundamental consumption utility as in Koszegi and Rabin (2006) and thus is neutral for decision making. To see how reference dependence helps make a bubble equilibrium more robust intuitively, loss aversion with expectation as the reference point creates a gap between the "reservation prices" from two sides of the market: In each equilibrium, the willingness to pay (WTP) for the marginal unit of risky asset is lower than the corresponding willingness to accept (WTA). This gap between WTP and WTA creates a "buffer" area that supports consistent trade behavior (including "no trade" outcome) invariant to small changes in the environment of the economy (beliefs, dividends, etc). Since such "consistent

[3] Among others, Kahneman (2011)'s observation that "no endowment effect is expected when owners view their goods as carriers of value for future exchanges, a widespread attitude in routine commerce and in financial markets" also supports the idea that in a trading scenario, expectations, instead of the status quo, can serve as an appropriate reference point - that is, a trader does not suffer from selling if she expects to sell, as suggested by the results of experiments in Ericson and Fuster (2011).

trade behavior" will not reveal any additional information to any agents, every agents' information remains the same regardless of the small parameter changes, and thus a bubble is robustly sustained in equilibrium. To be more specific, take the "no trade" scenario for example. The gap between WTP and WTA can induce strict loss from trade in the "no trade" scenario even in a competitive setting, and thus tiny perturbations in parameter values will not change the behavior of players, which further implies that the original information structure where a bubble exists can still be supported in the perturbed economy. Mathematically, by assuming reference dependent preferences, we transform a system of equalities to a system of weak inequalities, which is naturally more robust to deviations.

As a key step to make bubble robust, the gap between WTP and WTA can be generated in different ways. "Endowment effect" (Knetsch, 1989; Kahneman, Knetsch and Thaler, 1991) can definitely be used to create such a gap, and can also be interpreted by assuming loss averse preferences with the status quo as the reference point. In this paper, we choose rational expectation instead of status quo as the reference point, for mainly two reasons. First, we adhere to the rational expectations equilibrium (REE) concept following the standard literature on rational bubbles and it seems to us that using rational expectation as the reference point would potentially from this standpoint raise the fewest critiques by introducing least behavioral features to the classical model. Second, as has been mentioned earlier, empirical evidence supports the idea that expectations influence trading decisions (Hommes et al, 2008; Husler, Sornette and Hommes, 2013; Bao, Hommes and Makarewicz, 2017, among others) and some recent work suggests that expectation serves as a better reference point than status quo under trading environments in the laboratory (Ericson and Fuster, 2011).

Another essential part of our model distinct from the AMP model is in the definition of prices in the informational sense. We distinguish the explicit (underlying/ clearing) prices from the implicit (unobservable) prices. Although equivalent in most cases, the two notions depart from each other when the explicit price is not well-defined. Consider the popular double-oral auction (DOA thereafter) played in an elementary game theory class, where every agent is assigned a value of a typical good (for example a bar of chocolate). In each round, each seller (buyer) is asked to report privately a bid (ask) price and the auctioneer, after gathering all the information, decides on a market clearing price. If there exists some bid price (weakly) higher than some ask price[4], trades and clearing prices are declared, otherwise the auctioneer will just announce "silent market" and the game enters the next round or terminates without any common knowledge about the market price even if the market actually clears. Note that price being unobservable does not necessarily mean that the agent has no information about the price (for example in the "silent market" case a potential buyer (seller) knows for sure that the implicit market price is above (below) her bid (ask)

[4]This implies that there are positive (non-negative) gains from trades.

price). However, it does introduce some ambiguity about the clearing prices so that agents find it harder to distinguish different states through Bayesian updating, similar to how noise functions in the bubble literature (e.g. Doblas-Madrid, 2012).

Furthermore, it is worth-mentioning that most features (other than the non-robustness drawback) for bubbles in the AMP framework are preserved in our model. In particular, we show that the size of the robust bubble can approach the level of the highest dividend in the binary economy when agents attach high enough priors to those high-dividend states and it is also possible that the robust bubble occurs almost for sure in certain REE, similar to the result for non-robust bubbles with classical preferences (Lien, Zhang and Zheng, 2015).

For tractability and simplicity, in the main body of this paper we restrict our discussion to "binary economies" where each agent faces a binary prospect in the form $(p_1, q; p_2, 1-q)$ and "quasi-binary economies" which are perturbed versions of the corresponding binary economies. The results for a more general setting are provided in Appendix A.[5] Firstly, we show that the scenario of having an unobservable price is due to the lack of gains from trades, that is, no one wants to pay as high as the reservation price of anyone else, which follows the intuition of the DOA game mentioned earlier. Then, we characterize an agent's complete trading strategy as a function of (clearing) prices, where the WTP and WTA with respect to the status quo serve as the critical points. Furthermore, the existence of REE under any given initial information partition is shown and the robustness of bubble equilibria is examined.

To summarize, our work contributes to the literature by establishing a model for speculative bubbles that are robust to perturbations in parameters. We show that the same (and in most scenarios, a weaker version of the) condition sufficient to support (non-robust) rational bubbles in the classical AMP model is enough for robust bubbles in the reference-dependent settings, and the corresponding REEs in these two models share identical profiles of net trades. By allowing for the existence of unobservable prices, we generalize the AMP framework and develop new tools and concepts to conduct analysis in the no-trade scenarios with reference-dependent preferences. Another feature of our model preserved from the AMP framework, is that the bubble equilibrium can exhibit almost any feasible outcome by changing priors. For example, the bubble size can approach the level of the highest dividend, and it is also possible that the bubble can occur almost for sure in a REE. Our model also provides a first attempt to analyze a new setting in which the reference points are endogenously formed as agents' rational expectations in the asset market where prices are endogenously determined as well, while in the previous models by Koszegi and Rabin (2006, 2007) prices are taken as exogenously given.

[5]This restriction will not sacrifice intuition without loss of generality since in a more general setting in each range of the price where the sign of sensation is deterministic, the analysis actually resembles that in a binary economy. As a result, focusing on binary prospects simplifies our analysis only by avoiding the tedious calculation and discussions.

The remainder of the paper proceeds as follows. Section 2 summarizes the related literature. In Section 3, we describe the basic framework, introduce reference dependence and define the equilibrium. Section 4 shows the general equilibrium results, and Section 5 illustrates how a bubble occur in equilibrium and examinees its robustness in the simplest settings. We also discuss the the limit of bubble size, bubble frequency and the implications of our model to the real economy. Section 6 concludes.

2 Related Literature

Early theoretical work has provided sufficient conditions for the nonexistence of rational speculative bubbles. Milgrom and Stokey (1982) showed that under rational expectations the feasibility and individual rationality of equilibrium trade must be common knowledge, and thus an ex-ante Pareto optimal allocation of endowments of risky assets will lead to the ”no-trade” phenomenon even with heterogeneous beliefs and asymmetric information. Tirole (1982) further formulated the ideas into a definition of rational expectations equilibrium (REE), which has shown to be inconsistent with speculation (and thus speculative bubbles) unless traders have heterogeneous priors or can obtain insurance in the market. Since then, it has been well known in the literature that potential gains from trade and asymmetric information, together with short sales constraint, are necessary for a rational speculative bubble to exist in REE, as summarized in AMP (1993).

In the pioneering work of AMP (1993), whose framework we closely follow in this paper, the authors derive a finite-horizon model of rational bubbles with potential gains from trade, short sales constraint and asymmetric information, where a bubble is created due to higher order uncertainty, capturing the idea of the “greater fool”. To be more specific, in a T-period environment of two assets – one risky asset with no short sale constraints and one risk-free asset with short sale possibilities, dividends are realized in the last period and possess a stochastic distribution over possible states. Heterogeneous information is modeled by different initial partitions of the set of underlying states rather than the dividend structure over states. Also, REE is adopted as the equilibrium concept. Then, a (strong) bubble is observed under state ω in period t when the price $p_t(\omega)$ is strictly higher than any possible dividend any agent considers with positive probability to occur when the underlying state is ω. The intuition behind the existence of a rational bubble lies in the definition of speculative bubbles – even though all agents know that the asset is overvalued, they do not know that others also know this fact, which provides rational incentives to bid up the price of the risky asset over its fundamental value.

Following AMP (1993)’s framework, Conlon (2004) showed that rational bubbles are robust to higher (finite) order knowledge of the asset being overpriced, and Zheng (2014) showed that rational bubbles can be robust to both finite order knowledge of overpricing in the ”strong” sense (referring to the concept of ”finite order strong bubble” in that paper) and common knowledge of overpricing in the ”expected” sense (referring to the concept of ”common expected bubble” in that paper). However, the bubble equilibrium in these studies has a drawback of being fragile with small perturbations in parameters.[6] Our paper, by introducing reference-dependent preferences into the AMP framework, is able to create bubble equilibria robust to small perturbations in parameters.

Another related paper in the literature on rational bubbles is Lien, Zhang and Zheng (2015), which characterizes the information structure for the existence of bubbles in AMP framework, and studies the conditions that parameters on subjective beliefs and marginal utilities should satisfy for bubble size and bubble occurring frequency to achieve the maximum level. In our paper, we show that the main results on bubble size and bubble frequency in Lien, Zhang and Zheng (2015) still hold with the reference-dependent preferences.

Our paper is also related to the important strand of behavioral bubble literature that modifies the basic assumption of rationality by introducing some behavioral agents into the model. Abreu and Brunnermeier (2003) show the existence of bubbles through the interactions between behavioral traders and rational arbitrageurs, where those less-sophisticated traders over-optimistically expect permanent growth rate in fundamental value after a temporary shock and those rational arbitrageurs with hope to make profits by riding the bubble fueled by the behavioral agents may fail to do so due to the dynamic coordination problem. Scheinkman and Xiong (2003) formalize the overconfidence of agents such that they believe that their own information dominates others' and thus misjudge the possibilities of selling the bubble assets in time and engender the bubble. Harras and Sornette (2011) show that the myopic investors' attempt to adapt their trading strategy to the current market regime will paradoxically lead to a dramatic volatility in prices, including bubbles and crashes. Huang, Zheng and Chia (2013) characterize how bubbles arise gradually while crashes happen suddenly in a deterministic heterogenous agent model (Huang, Zheng and Chia, 2010; Huang and Zheng, 2012) where heterogeneity is in terms of expectations of the fundamental value by two types of traders. Fundamentalists have constant expectations while the chartists update expectations periodically based on observations of previous prices. All these studies mentioned above rely on the presence of some traders with either behavioral trading strategies or behavioral beliefs in order to support the existence of bubbles, while the behavioral feature in our model stems only from the reference-dependent preferences, accompanied by Bayesian beliefs and rational strategies.

A third category of literature related to our work is the theoretical studies on reference-dependent preferences and reference point formation. Since Tversky (1976)'s work on prospect theory, var-

[6] Zheng (2014) shows that the framework of AMP, though non-robust in general, is robust to a class of symmetric perturbations to beliefs and another class of symmetric perturbations to dividends. Conlon and Zheng (2013) introduce a continuum of states into the AMP framework and makes the bubble equilibrium more robust.

ious theories have been produced to explain the reference dependent phenomena. Among those well-established models, Koszegi and Rabin (2006) build a testable theory to capture the key feature of loss aversion in the reference-dependent scenarios and the reference point is endogenously determined as the rational expectations.[7] Koszegi and Rabin (2007) extend their previous work to study preferences with monetary risk. The reference-dependent utility has also been incorporated into the theory literature in other fields like industrial organization and auctions: Such studies include Heidhues and Koszegi (2014), in which a risk-neutral profit-maximizing monopolist chooses the optimal pricing strategy by manipulating loss-averse customers due to their reference-dependent preferences, and Ahmad (2015), which incorporates reference dependence with expectations as the endogenous reference point into the classical framework of auctions and predicts both overbidding and underbidding in equilibrium compared to the standard risk-neutral Nash equilibrium. From this perspective, our paper contributes to this literature by incorporating reference dependence into the classical framework of general equilibrium with incomplete information.

Last but not the least, it is worthwhile to compare our work with that of Doblas-Madrid (2012), which also presents robust bubbles with potential gains from trade, asymmetric information and short sales constraint. However, the frameworks and focuses are quite different. Following the framework in Abreu and Brunnermeier (2003), where the bubble equilibrium is already robust, Doblas-Madrid (2012) introduces multi-dimensional uncertainty into the model, aiming to make a fully rational model of bubbles by taking away the assumption of the behavioral agents, which was originally the crucial component in Abreu and Brunnermeier (2003). By contrast, our paper is based on the AMP (1993) framework of fully rational bubbles, trying to generate robust bubble equilibria by introducing behavioral features into the model. In short, Doblas-Madrid (2012)'s focus is to make an existing robust bubble model rational, while ours is to make an existing rational bubble model robust.

To summarize how this paper fits in the literature, our main purpose is to contribute to the theory of robust speculative bubbles by developing a reference-dependent model following AMP (1993), in response to the major critiques to previous studies on the non-robustness issue. By incorporating this behavioral feature into the model and using expectations as agents' reference points, we show that the bubbles become quite robust to parameter perturbations and most of the results about bubble size and bubble frequency under the classical AMP setting are preserved with reference dependence. From a broader perspective, our work also contributes to research on improving classical models by incorporating more realistic behavioral features, with higher explanatory power and more broadly applicable scope.

[7] As recent evidence from the lab, Ericson and Fuster (2011) present the experiments where subjects with a lower exogenous probability of being able to trade their items (and who therefore expect to keep it with a higher probability) are less likely to choose to trade when such an opportunity arises, implying that the reference point is (at least partly) determined by the expectations instead of the status quo.

3 Basic Framework

Following the framework of AMP (1993), we consider the financial market as a pure exchange economy, with I agents ($i = 1,2,\cdots,I$, $I \geqslant 2$), T periods ($t = 1,2,\cdots,T$, $T \geqslant 3$), N states ($\omega \in \Omega$, $|\Omega| = N$) and two assets: one risk-free (money) and one risky with short sales constraints imposed on the latter.[8] Agent i holds endowment of m_i units of money and e_i units of risky assets. For simplicity, assume that there is no discount between any two periods and each share of the risky asset will pay a state-dependent dividend $d(\omega) : \Omega \longrightarrow \mathbb{R}$ only at the end of period T, where the structure of $d(\omega)$ is common knowledge.

In each period t ($1 \leqslant t \leqslant T$) with the underlying state $\omega \in \Omega$, agent i chooses whether to change the position of the risky asset at a state-dependent price $p_t(\omega)$. Denote her (net) trade under this circumstance as $x_{it}(\omega)$. For ease of notation, in state ω, write $x_i(\omega) = (x_{i1}(\omega),\cdots,x_{iT}(\omega))$ as the trading strategy of agent i across all periods, $x_t(\omega) = (x_{1t}(\omega),\cdots,x_{It}(\omega))$ as the net trades of all agents in period t, $X(\omega) = (x_1(\omega),\cdots,x_I(\omega))$ as the trading strategy profile across all periods, and $P(\omega) = (p_1(\omega),\cdots,p_T(\omega))$ as the prices of the risky asset across all periods.

Assume agent i consumes all her wealth only after the dividends are realized, that is, she possesses a final consumption level: $y_i(\omega,P,x_i) = m_i + e_i \cdot d(\omega) + \sum_{t=1}^{T} x_{it}(\omega) \cdot (d(\omega) - p_t(\omega))$. Let $u(\cdot)$ be the utility function of every agent. Each agent i has a subjective belief about the probability distribution of the states, denoted by $\pi_i(\omega), \omega \in \Omega$. Without loss of generality, assume that $\pi_i(\omega) > 0$, $\forall i = 1,2,\cdots,I$, $\forall \omega \in \Omega$, and it is possible that $\pi_i(\omega) \neq \pi_j(\omega), \forall i \neq j$.[9] Notice that when deciding on the trading strategy, an agent is not necessarily sure about the underlying state and thus what she really cares about is to maximize the ex-ante form of her utility.

Following a traditional methodology to model incomplete information (Milgrom and Stokey, 1982; Samuelson, 2004), we represent agents' information using partitions, as is commonly adopted in the bubble literature (AMP, 1993; Conlon, 2004; Zheng, 2014, among many others). Denote agent i's exogenous information about the states in period t as S_{it}, a partition of Ω, and the information set including ω as $s_{it}(\omega)$. That is, $s_{it}(\omega) \in S_{it}$ such that $\omega \in s_{it}(\omega)$ and $s_{it}(\omega)$ consists of all states that agent i believes possible at the beginning of period t when ω is realized. Notice that by observing the market prices and the net trades of other agents, agent i will update her information in the sense that states with different prices or net trades can be distinguished by her. Formally, we borrow the notation from Zheng (2014) and define $s_{it}^{PX}(\omega)$ as the price-and-trade-refined

[8] Without further illustration, the dividends and prices referred to below are all measured in terms of money - the risk-free asset.

[9] In AMP (1993), it is shown that either heterogeneous subjective priors or heterogeneous marginal utilities can lead to potential gains from trade, which is a necessary condition for bubble to exist. Here, we take the former approach and assume that all agents have the same utility functions for simplicity.

information set such that

$$\forall \omega \in \Omega,\ s_{it}^{PX}(\omega) = s_{it}(\omega) \cap \left\{\omega' | p_{t'}(\omega') = p_{t'}(\omega'), \forall t' \leqslant t\right\} \cap \left\{\omega' | x_{t'}(\omega') = x_{t'}(\omega'), \forall t' \leqslant t\right\}.$$

A direct observation from the above definition is that $s_{it}^{PX}(\omega) \subset s_{it}(\omega)$. In fact, $s_{it}^{PX}(\omega)$ represents the set of states that agent i believes possible at the end of period t when ω is realized, and $S_{it}^{PX} \equiv \bigcup_{\omega \in \Omega} \left\{s_{it}^{PX}(\omega)\right\}$ is agent i's information partition in period t after taking into account the endogenous signals from the price and the net trades of all agents.[10] Although every agent has private information, we follow the literature and assume that the information partitions are common knowledge among agents.

Meanwhile, we also assume that agents have "perfect memory" (Zheng, 2014), such that

$$\forall \omega \in \Omega,\ \ \forall i = 1, 2, \cdots, I,\ \ \forall t > t',\ \ s_{it}(\omega) \subset s_{it'}(\omega).$$ [11]

For simplicity, we suppose that all asymmetry and incompleteness of information is removed in the last period T. That is, $\forall \omega \in \Omega, s_{it}^{PX}(\omega) = s_{it}(\omega) = \left\{\omega\right\}$. This implies that in period T it is common knowledge that the fundamental value of the risky asset is $d(\omega)$ and agents value the risky asset in this way. Thus, we have $\forall \omega \in \Omega, p_T(\omega) = d(\omega)$.

3.1 Risk Attitudes and Reference Dependence

In previous literature, the utility functions are either assumed to be piecewise linear (AMP, 1993; Conlon, 2015), or risk-neutral (Conlon, 2004; Zheng, 2014), which is regarded as a reasonable approximation if the volume and the risk of the trading behavior in the market under consideration are both small relative to an agent's overall economic activities. This simplification does provide great tractability for the analysis by avoiding the trouble of dealing with diminishing marginal utilities, but, to some extent, results in non-robustness.

Intuitively speaking, creating a gap between WTP and WTA is the key for bubbles to persist regardless of tiny changes in parameters, as equilibrium conditions are now in the form of inequalities instead of equalities, the former of which can be robust to small parameter perturbations. However, it is worth noting that a gap between WTP and WTA in theory can be produced by assuming diminishing sensitivity of wealth alone, as shown in the following example. Consider that an agent with wealth level w and classical preferences, who currently holds a unit of a risky asset

[10] As noted in AMP (1993), Conlon (2004) and Zheng (2014), in a rational expectations equilibrium (REE), the information based on which an agent makes decisions should be S_{it}^{PX} instead of S_{it}. We will come back to this when defining a REE.

[11] The assumption of perfect memory implies that the exogenous information can only be refined along with time, which is essential to the non-existence result of strong bubbles with common knowledge in Zheng (2014). This assumptiom makes good sense if agents are considered rational.

$(p,d;1-p,0)$, is indifferent among 3 options: (1) staying with the status quo or (2) paying WTP to buy one more unit of the asset or (3) being paid by WTA to sell that unit of asset at hand. This implies $u(w+WTA) = pu(w+d)+(1-p)u(w) = pu(w+2d-WTP)+(1-p)u(w-WTP)$, and by strictly diminishing sensitivity (strict concavity of $u(\cdot)$) we will produce $WTP < WTA$. Given the above result, it is natural to ask why we bother to bring the behavioral feature of reference-dependence into the current model, instead of simply assuming concave utilities.

Reconsider the above example. Suppose now for the agent, $m = 100,\ d = 1,\ p = \frac{1}{2},\ u(x) = \sqrt{x}$, then it is easy to show that $WTA - WTP = 0.0012 \ll 1 = d$. This result implies that WTP and WTA will become sufficiently close as long as the fundamental value of the risky asset is small enough relative to the life-time wealth level. To make it more transparent, we adopt an explicit illustration of agent i's endowment of money. Consider that agent i will earn wages in every period t and then choose whether to consume right away or to save for retirement after the end of period T. Assume realistically that the size of the dividend of the risky asset is not comparable to the wage level, which implies that when making the decision on optimal consumptions from period 1 to T, the agent takes the gain/loss on the risky assets as irrelevant. Also, borrowing the risk-free asset (i.e. money) is assumed to be allowed at the interest rate of 0 (only for simplicity) as long as the agent does not die with debt. Denote the aggregate saving of wages for retirement by m_i, which is defined as agent's endowment of money, where $m_i \gg max_{\omega\in\Omega}(d(\omega))$. Thus, the decision concerning the activities in the financial market is only of small stakes. Rabin (2001) has shown that the concavity of utility functions under expected-utility theory does not serve as an plausible explanation for appreciable risk-aversion over modest stakes since any utility representation that does not predict absurdly counter-intuitive risk aversion over large stakes (for example, loss of \$2000 outweighs gain of infinity) will predict negligible risk aversion under modest risk. The fact that the commonly used differentiable utility functions can only display second-order risk aversion and local linearity calls for a non-stationary kink in the functional form to account for different attitudes over small positive shocks versus negative ones, which coincides with the feature of loss aversion with respect to the reference point. Therefore, following the KR approach, in this paper we assume that every agent has a reference-dependent utility function:

$$u(y|r) = m(y) + \mu(m(y) - m(r)),$$

where y denotes the consumption level, r denotes the reference point of consumption,[12] $m(\cdot)$ is the standard monotonic differentiable utility function for consumption, and $\mu(\cdot)$ is the gain-and-loss utility function. Note that since the endowment of wealth is much greater than the possible fundamental value of one unit of risky asset and $m(\cdot)$ is locally linear, it makes sense to assume

[12] Here we use the recent expectations as the reference point, as in Koszegi and Rabin (2006, 2007).

that the marginal utility of consumption is constant in the domain of interest.

Normalizing $m(y)$ to y, the utility function reduces to $u(y|r) = y + \mu(y-r)$. As for the gain-and-loss function $\mu(\cdot)$, we assume $A0-A4$ the same as in Koszegi and Rabin (2006). For $A3$: $\mu''(x) \leqslant 0, \forall x > 0$ and $\mu''(x) \geqslant 0, \forall x < 0$, note that (1) the first-order risk-aversion will dominate the second-order risk-aversion over small stakes and (2) the curvature of consumption utility function $m(\cdot)$ will matter when stakes are large. To simplify our analysis without loss of intuition, given that the risky decisions in our framework are over small stakes, we focus on the special case of $A3$ denoted as $A3'$: $\mu''(x) = 0, \forall x \neq 0$. This implies $\mu'_+(x) \equiv \eta > 0$, $\mu'_-(x) \equiv \lambda\eta$ where $\lambda > 1$ and

$$\mu(x) = \begin{cases} \eta x & x \geqslant 0; \\ \lambda\eta x & x < 0. \end{cases} \tag{1}$$

We follow the standard assumption that preferences are linear in probabilities, consistent with the ideas of "PEEMs" in Rabin (2013), since the feature of loss-aversion has been already incorporated into the part of gain-and-loss utilities. If the consumption and the reference point take on some probabilistic distributions of F and G respectively, an agent's expected utility function should be $U(F|G) = \int_y \int_r u(y|r)dF(y)dG(r)$.

3.2 Notion of Equilibrium

The basic concept of rational expectations equilibrium for bubbles originated from AMP (1993), with a formal characterization in Zheng (2014).

Definition 1 (Information Feasibility) *Agent i's net trades x_i are information feasible if in each period t, x_{it} is measurable with respect to player i's price-and-trade-refined information.*

Definition 2 (Short Sales Constraint) *Agent i's net trades x_i satisfy short sales constraint if in each period t and in each state ω, agent i's holdings of the risky assets are nonnegative.*

Denote the set of all such x_i's that are information feasible and satisfy short sales constraint by $F_i(e_i, P, S_i)$, and we can define the notion of equilibrium for our framework.

Definition 3 (Rational Expectations Equilibrium) $(X, P) \in \mathbb{Z}^{INT} \times \mathbb{R}_+^{NT}$ *is a rational expectations equilibrium (REE) if*

(C1) Information Feasibility and Short Sale Constraint: $\forall i = 1, 2, \cdots I, x_i \in F_i(e_i, P, S_i)$;

(C2) Market Clearing Condition: $\forall t = 1, 2, \cdots T$, $\forall \omega \in \Omega$, $\sum_{i=1}^{I} x_{it}(\omega) = 0$;

(C3) Upper Bound of Price Information: $\forall t = 1, 2, \cdots T$, $p_t(\omega)$ *is measurable with respect to the join of* $s_{it}(\omega)$, $i = 1, 2, \cdots I$, *denoted by* $j_t(\omega)$;

(C4) Utility Maximization: each agent's net trades are optimal, given her information, prices, short sale constraints and correct belief about others' strategies. Formally, $\forall i = 1,2,\cdots I, x_i \in$ *arg max*$_{x'_i \in F_i(e_i,P,S_i)} \sum_{\omega\in\Omega} \pi(\omega) u_i(y_i(\omega,P,x'_i))$.

3.2.1 A Tractable Simplification of Utility Maximization in *REE*

The form of the utility function of agent i, $u_i(y_i(\omega,P,x_i)) = u_i(w_i + e_i \cdot d(\omega) + \sum_{t=1}^{T} x_{it}(\omega)\cdot(d(\omega) - p_t(\omega)))$, naturally provides the intuition that the asset price in period t, p_t, should be compared with the size of final dividend when the agent decides her optimal trading strategy. If agent i can get additional capital gains apart from the possible underlying final dividends in period t, then in prior periods, to decide on current net trades to maximize final consumption, she will not use the final dividend distribution as the comparison level for the current price. Instead, she rationally expects that in a competitive market, she can get any feasible position in period t with potential gains. Provided that the capital gains will add up as we trace backwards from period T when price equals fundamental value across all states, the price structure in period $t+1$ given the price-and-trade-refined information $s_{it}^{PX}(\omega)$ should form agent i's opportunity cost of selling the risky asset in period t facing the underlying state ω. Accordingly, under the assumption of rational expectations and perfect memory, $(C4)$ is equivalent to maximizing per-period utility given the rational expectations on prices in the next period formed from the information in the current period.

Formally, consider with the starting position $(m_{it}(\omega), z_{i(t-1)}(\omega) \equiv e_i + \sum_{\tau=1}^{t-1} x_{i\tau}(\omega))$ in period t, agent i should choose $x_{it}(\omega) \in D_{it}(\omega) \equiv \{x \in \mathbb{Z} : x \geqslant -z_{i(t-1)}(\omega)\}$ based on the price profile $P = \{P(\omega) : \omega \in \Omega\}$.[13] Denote agent i's expected utility in period t at state ω by trading y units of risky assets while expecting to trade x units, by $V_{it\omega}(y|x)$, where $x, y \in \mathbb{Z}$.[14] Note that $x_{it}(\omega)$ maximizes utility for agent i in period t at state ω if and only if

$$\textit{(C5)} \qquad x_{it}(\omega) \in argmax_{x\in\mathbb{Z}}\, V_{it\omega}(x|x_{it}(\omega)), \;\; s.t. \;\; x \geqslant -z_{i(t-1)}(\omega).$$

The intuition of *(C5)* follows the notion of *personal equilibrium (PE)* in Koszegi and Rabin (2006) in the sense that the trading plan is valid for an agent with rational expectations if and only if she has no incentive to deviate from the plan when she actually faces the corresponding choice, otherwise she will ex-ante deviate from it by modifying her plan. From this perspective,

[13] Rigorously speaking, agent i's information about the price in state ω in period t' ($t' > t$) should be the expected price $E_{it}[p_{t'}(\omega)|s_{it}^{PX}(\omega)]$. However, in *REE*, all agents (correctly) predict the prices such that $E_{it}[p_{t'}(\omega)|s_{it}^{PX}(\omega)] = p_{t'}(\omega)$.

[14] That is, $V_{it\omega}(y|x) = \frac{1}{(\sum_{\omega'\in s_{it}^{PX}(\omega)} \pi(\omega'))^2} \sum_{\omega'\in s_{it}^{PX}(\omega)} \sum_{\omega''\in s_{it}^{PX}(\omega)} \pi_i(\omega')\pi_i(\omega'') u((m_{it}(\omega) + (z_{i(t-1)}(\omega)+y)p_{t+1}(\omega') - yp_t(\omega))|(m_{it}(\omega) + (z_{i(t-1)}(\omega)+x)p_{t+1}(\omega') - xp_t(\omega)))$. This can be interpreted as the utility function faced by i in period t at state ω. If possibly $p_{t+1}(\omega')$ is not well-defined (See Section 3.2.2 for details), then replace it by $p_{t'}$ such that t' is the first period after t when $p(\omega)$ is observable. Notice that firstly t' must exist since $p_T(\omega) = d(\omega)$. Also, this will not influence our analysis since when prices are not observable, net trades must be 0.

our notion of *REE* is an extended notion of *PE* that incorporates the interactions of agents and formation of prices. Intuitively, every agent correctly predicts the price distribution, which is taken as exogenous given, as well as her reaction to the prices to form a PE and then those individually rational plans are collected by the ”invisible hand” to determine, based on the conditions of market clearing and information feasibility, the equilibrium market price and thus the *REE*. From now on we will use *(C5)* as a substitute of *(C4)*.

For ease of notation, we define the WTP (WTA) for the kth($k>0$) unit of risky asset as $WTP^k_{it}(\omega|x_i)(WTA^k_{it}(\omega|x_i))$[15] with respect to the reference point as net trading x_i units of risky asset, that is,

$$\frac{1}{(\sum\pi_i(\omega'))^2}\sum\sum\pi_i(\omega')\pi_i(\omega'')u((m_{it}(\omega)+(z_{i(t-1)}(\omega)+k)p_{t+1}(\omega')-(k-1)p_t(\omega)-WTP^k_{it}(\omega|x_i))|(m_{it}(\omega)+(z_{i(t-1)}(\omega)+x_i)p_{t+1}(\omega'')-x_ip_t(\omega)))=V_{it\omega}(k-1|x_i)$$

$$\frac{1}{(\sum\pi_i(\omega'))^2}\sum\sum\pi_i(\omega')\pi_i(\omega'')u(m_{it}(\omega)+(z_{i(t-1)}(\omega)-k)p_{t+1}(\omega')+(k-1)p_t(\omega)+WTA^k_{it}(\omega|x_i)|(m_{it}(\omega)+(z_{i(t-1)}(\omega)+x_i)p_{t+1}(\omega'')-x_ip_t(\omega)))=V_{it\omega}(-k+1|x_i).$$ [16]

3.2.2 Unobservable Prices

In a competitive equilibrium, the price vector must **exist** as an ”invisible hand” to help allocate scarce resources. Even if there is no net trade, it should be the case that $V_{it\omega}(0|0)\geqslant V_{it\omega}(1|0)$ and $V_{it\omega}(0|0)\geqslant V_{it\omega}(-1|0)$ where prices are necessary for valuation. However, this does not imply that a specific price can be **observed** by the agents. Consider the case in the process of DOA that any bid price is strictly lower than any ask price and then the auctioneer will not declare any deal or price, it is reasonable that a typical agent can only infer an interval which the clearing price belongs to, instead of actually observing it. Based on this understanding, we can define unobservable prices in terms of information structure and trade incentive:

Definition 4 (Unobservable Prices) *In a REE, the price in period t with underlying state ω is* ***unobservable (or not well-defined or implicit), denoted as*** $p^{PX}_{it}(\omega)=\emptyset$*, if there is no net trade ($x_t(\omega)=0$) and under the actual clearing price there is a strict loss from any potential trade among the agents with the reference point at the no-trade status quo.*

In other words, a no-trade REE price is unobservable if there do not exist two agents such that one strictly (weakly) prefers buying to the status quo and the other weakly (strictly) prefers selling to the status quo, given the reference point set as the status quo. Here the symbol $\emptyset$ may be a little bit confusing, but keep in mind that we use $\emptyset$ only in the informational sense and it simply means ”price being not observable by any agent”.

Correspondingly, prices are observable (or well-defined or explicit) if (1) “$x_t(\omega)\neq 0$” or (2)

[15] For simplicity, we refer to $WTP^1_{it}(\omega)$ as $WTP^1_{it}(\omega|0)$, to represent the case with reference point as the no-trade status quo.

[16] Here $\sum$ represents $\sum_{\omega'\in s^{PX}_{it}(\omega)}$ and $\sum\sum$ represents $\sum_{\omega'\in s^{PX}_{it}(\omega)}\sum_{\omega''\in s^{PX}_{it}(\omega)}$.

"$x_t(\omega) = 0$ but at least two agents have weak incentive to trade between them".[17] Recall from the DOA scenario that the auctioneer will definitely announce net trades along with the equilibrium price in case (1) and will possibly do so in case (2) due to the arbitrary tie-breaking rule.[18] By allowing the price to be unobservable, we are able to deal with the technical difficulty in the equilibrium analysis with incomplete information. Without any further modification in the original definitions of information feasibility and the corresponding *REE*, we naturally extend these concepts to be applicable to broader scenarios, with the previous studies under the AMP framework as a special case in our model. It can be shown that when all agents have classical preferences, in the AMP framework, the equilibrium prices are always observable even where there is no-trade. This result is characterized in the following Claim and we defer the proof to Appendix B.

Claim 1 *With classical reference-independent utility functions, prices are always observable.*

3.3 A Generalized Notion of Strong Bubbles

Since clearing prices can be unobservable, the definition of strong bubbles under the classical AMP framework should be slightly modified accordingly.

Definition 5 (Strong Bubble) $\omega \in \Omega$ *is said to exhibit a* ***first-order strong bubble*** *in period t in a REE (X,P) if either* (1) $p_t(\omega) = \emptyset$ *and* $WTP^1_{it}(\omega) > max_{\omega \in s^{PX}_{it}(\omega)} d(\omega)$, $\forall i$; *or* (2) $p_t(\omega) > max_{\omega \in s^{PX}_{it}(\omega)} d(\omega)$, $\forall i$.

A bubble occurs when it is mutual knowledge that the asset is "over-valued" compared to the fundamental value. The term "over-valued" in the classical AMP model means that the current price of the asset is higher than the asset's any possible future value (condition (2) in Definition 5), while in our setup "over-valued" also refers to the case where every agent's willingness to pay for the asset is higher than the asset's any possible future value when the price of the asset is not well-defined in the no-trade scenario (condition (1) in Definition 5).

The key intuition for such a bubble to occur in *REE* is as follows. Although every agent knows for sure that the asset is "over-valued", they do not know whether other agents know that the asset is "over-valued", hence no agent has incentive to ride the bubble at present because everyone rationally expects that she may benefit more if waiting until a later period to sell the "overvalued" asset to a "less-informed" agent, which is called "a greater fool" in Conlon (2004) and Doblas-Madrid (2012). While on the equilibrium path in some state an agent does successfully find such a "greater fool" and makes profit by selling the "over-valued" asset, in some other state every agent ends up with holding the "over-valued" asset at hand when the bubble crashes. The chance of

[17] Later we will refer to the case of $x_t(\omega) = 0$ as a silent market in period t in state ω.

[18] An alternative way to understand that prices are well-defined in case (2) is as follows: Under the specified underlying clearing price, there can be a net trade which makes the trading agents as well off as the no-trade case. Consider that there are numerous replicas of the agents in the market, then such a net trade may happen with a positive probability, thus it is natural to define the price in the no-trade case by the price with trade.

successful speculation and the chance of failure of escaping balances in a way such that agents ex ante are willing to hold the asset even though they know it is "over-valued".

4 Equilibrium Analysis

4.1 Information Partition as a Singleton

We first analyze the case with certainty and summarize the findings below. It is easy to understand that with complete information the result with reference-dependent preferences is totally consistent with the classical predictions..

Proposition 1 *Given t' and ω, if $|s^{PX}_{it'}(\omega)| = 1$, then $\forall t$ such that $t' \leqslant t \leqslant T$, agent i's valuation of the risky asset in period t in state ω is $p_{t+1}(\omega)$, with $p_{T+1}(\omega) \equiv d(\omega)$. Furthermore, if $p_t(\omega) > p_{t+1}(\omega)$, i should sell all risky assets she holds; if $p_t(\omega) = p_{t+1}(\omega)$, i is indifferent among all positions; if $p_t(\omega) < p_{t+1}(\omega)$, i should buy as many units of risky assets as she can.*

Intuitively, once a agent knows for sure about the underlying state, there is no risk (measurable uncertainty) for her in all following periods. To see this, notice that first-order risk aversion only exists in prospects involving both gains and losses and the second-order risk aversion is assumed away by $A3'$. Thus the agent knows for sure that the exact opportunity cost of trading is the asset price in the next period, and how she will actually trade depends on the relative level of the current price and the price in the next period. The proof to and the results in Proposition 1 immediately imply the following corollaries.

Corollary 1 *Given t and ω, if $\forall\ \omega',\ \omega'' \in s^{PX}_{it}(\omega)$, $p_{t+1}(\omega') = p_{t+1}(\omega'')$, all results in Proposition 1 hold.*

Corollary 2 *$\forall \omega$, $\forall t$, prices are always well-defined in period t in state ω if $\forall i$, $|s^{PX}_{it}(\omega)| = 1$.*

4.2 Relationship between WTP and WTA at REE

Recall from Section 3.1 that for an individual with classical risk-averse preferences, $WTA > WTP$ always hold. For reference-dependent utility functions (not necessarily concave everywhere) we have a weaker result that at least holds in equilibrium, and this is enough for resolving the robustness issue.

Proposition 2 *In a $REE(X,P)$, $\forall i = 1,2,\cdots I$, $\forall \omega \in \Omega$, $\forall 1 \leqslant t \leqslant T$, if $z_{it} > 0$, then $WTA^1_{it}(\omega|x_{it}(\omega)) \geqslant WTP^1_{it}(\omega|x_{it}(\omega))$.*

Intuitively, since a utility-maximizing agent with a positive holding of the asset can freely choose to buy or sell at least one more unit of the risky asset with respect to her current holding but actually does not do so in equilibrium, it must be the case that she values an additional unit of the asset no more than the prevailing price and she values the last unit of the asset she has purchased no less than the prevailing price.

4.3 Equivalent Conditions for Unobservable Prices

Before discussing about the price formation rule, we would like to impose an extra assumption to simply our analysis. With a kink at the reference point, the gain-loss utility function does not exhibit a uniform and smooth shape, and this can potentially cause very tedious discussions about all possible clearing prices in order to to deliver a complete characterization of the trading strategy as a function of price.[19] For simplicity, but without loss of generality, we conduct our analysis in the environment of so called "binary economies" in the following sections, and show that our results hold in general settings in Appendix A.

Definition 6 (Binary Economy) *The exchange economy is* ***binary*** *if* $\forall i = 1,2,\cdots I$, $\forall \omega \in \Omega$, $\forall 1 \leqslant t \leqslant T$, *agent i faces the binary prospect* $(p^{1*}_{i(t+1)\omega}, q_{it\omega}; p^{2*}_{i(t+1)\omega}, 1-q_{it\omega})$, *where* $p^{j*}_{i(t+1)\omega} (j = 1,2)$ *denote two possible prices in period* $t+1$ *under* $s^{PX}_{it}(\omega)$ *and* $p^{1*}_{i(t+1)\omega} \geqslant p^{2*}_{i(t+1)\omega}$.

Notice that the assumption of binary economy is made only to simply the analysis involving gain-loss utility functions. As for a more complicated economy, in any range of prices where the sign of each mental sensation is deterministic, the analyses resemble those described in the binary one except with more tedious discussions and heavier workload. In this sense, we can safely claim that the assumption of binary economy will not sacrifice any important insight in the analysis and most results can be easily extended to the general case.

Proposition 3 *Given t and* ω, *suppose agent i faces the binary prospect* $(p^{1*}_{i(t+1)\omega}, q_{it\omega}; p^{2*}_{i(t+1)\omega}, 1-q_{it\omega})$, $\forall i = 1,2\cdots I$. *In any REE, the price* $p_t(\omega)$ *is unobservable if and only if* $WTP^1_{it}(\omega) < WTA^1_{jt}(\omega)$, $\forall i \neq j$.

Recall that $WTA^1_{it}(\omega)$ ($WTP^1_{it}(\omega)$) is agent i's willingness to pay (willingness to accept) for the first unit of risky asset with respect to the status quo (as her reference point). The intuition of Proposition 3 is as follows: When everyone expects that the maximum they are willing to pay for an additional unit of the risky asset strictly falls short of anyone else's reservation prices, there will be a silent market with no observable prices. Accordingly, for binary prospects (and also for quasi-binary prospects, as will be shown later), to show that price is unobservable in period t at state ω, it suffices to show that $WTP^1_{it}(\omega) < WTA^1_{jt}(\omega)$, $\forall i \neq j$, which is a more tractable way in terms of calculation. Specifically, if we have $WTP^1_{it}(\omega) < WTA^1_{jt}(\omega)$, $\forall i,j$, then all the potential clearing prices will form an interval $[max_i WTP^1_{it}(\omega),\ min_i WTA^1_{it}(\omega)]$.

[19]Notice that this type of complexity only exists in the overall analysis aiming at obtaining general conclusions on REE. If we are only interested in verifying a specific REE, the computational work is manageable since gains and losses can be easily distinguished with certainty.

Now we are ready to explain why bubbles are robust in the AMP framework with reference-dependent preferences. Note that $WTP^1_{it}(\omega)$ and $WTA^1_{it}(\omega)$ are continuous (not necessarily smooth though) in $p_{t+1}(\omega)$ and $q_{it\omega}$, both of which are continuous in dividends and priors. Thus $WTP^1_{it}(\omega)$ and $WTA^1_{it}(\omega)$ are continuous in dividends and priors, implying that with sufficiently small perturbations in those parameters, $max_i\, WTP^{1'}_{it}(\omega) < min_i\, WTA^{1'}_{it}(\omega)$ still hold. Thus, in the information set including the underlying state, an agent in different environments of parameters may evaluate the risky asset differently, but may actually observe the same silent market with unobservable prices. To be more specific, every agent only knows for sure that (1) everyone wants to pay less than the minimum of other agents' reservation prices, and (2) the underlying (unobservable) clearing price lies between her own WTP and WTA, and she learns nothing new about the underlying state compared to the information she has in the environment of original parameters. Thus the previous whole structure can still be supported by the prices, and the REE with a speculative strong bubble can persist. To put it in a simply way, the positive gap between WTP and WTA functions in a way that makes agents insensitive to the difference in underlying prices when they are unobservable.

4.4 Trading Strategy as the Best Response

By **Proposition 3**, we know that if in period t at state ω the silent market (no-trade scenario) can never appear in any REE, then it must be the case that the price is well-defined and weakly lower than the willingness to pay with respect to status quo for the buyers and weakly higher than the willingness to accept with respect to status quo for the sellers, with at least one inequality relation holding strictly. Since it is assumed that the wealth of an agent is much larger than the unit price of the asset and there are short-sale constraints, to satisfy the conditions of market clearing and utility maximization, a buyer has to be indifferent between whether to buy one more unit of the asset or not in the equilibrium and the seller will probably reach the binding short-sale constraint.

Proposition 4 *In a binary economy, Given t and* ω*, if there is no REE where* $x_t(\omega) = 0$*, then for any agent i, either* (1) $p_t(\omega) = \frac{(1+\eta)q_{it\omega}p^{1*}_{i(t+1)\omega}+(1+\eta\lambda)(1-q_{it\omega})p^{2*}_{i(t+1)\omega}}{(1+\eta)q_{it\omega}+(1+\eta\lambda)(1-q_{it\omega})}$ *or* (2) $-z_{i(t-1)}(\omega)$ *satisfies (C5), that is,* $-z_{i(t-1)}(\omega) \in argmax_{x\in\mathbb{Z},x\geqslant -z_{i(t-1)(\omega)}}\, V_{it\omega}(x|-z_{i(t-1)}(\omega))$.

Notice that there can be multiple equilibria in a binary economy with a given information structure. **Proposition 4** only specifies the result in the case where in any REE we must have $x_t(\omega) \neq 0$, but is silent when faced with the possibility that $x_t(\omega) = 0$ and $x_t(\omega) \neq 0$ can simultaneously appear under the same settings. It is also worth noting that the seemingly complicated expression on the right hand side of condition (1) in **Proposition 4** is in fact i's WTP with respect to the status quo, that is, $WTP^1_{it}(\omega) = \frac{(1+\eta)q_{it\omega}p^{1*}_{i(t+1)\omega}+(1+\eta\lambda)(1-q_{it\omega})p^{2*}_{i(t+1)\omega}}{(1+\eta)q_{it\omega}+(1+\eta\lambda)(1-q_{it\omega})}$. Similarly, we can derive that

$WTA^1_{it}(\omega) = q_{it\omega} p^{1*}_{i(t+1)\omega} + (1 - q_{it\omega}) p^{2*}_{i(t+1)\omega}$.[20] Therefore, we can specify an agent's trading s-trategy as a function of the prevailing market price, compared to the agent's WTP and WTA.

Claim 2 (Trading Strategy in a Binary Economy) *In period t at state ω, facing the binary prospect $(p^{1*}_{i(t+1)\omega}, q_{it\omega}; p^{2*}_{i(t+1)\omega}, 1 - q_{it\omega})$, where $p^{1*}_{i(t+1)\omega} > p^{2*}_{i(t+1)\omega}$, agent i's trading strategy should be the following:*

(1) If $p_t(\omega) > WTA^1_{it}(\omega)$, then $x_{it}(\omega) = -z_{i(t-1)}(\omega)$;

(2) If $p_t(\omega) = WTA^1_{it}(\omega)$, then $x_{it}(\omega) \in \mathbb{Z} \bigcup \left[-z_{i(t-1)}(\omega), +\infty\right]$. In fact, for any given $x_{it}(\omega) \in \mathbb{Z}$, $argmax_{x \in \mathbb{Z}, x \geqslant -z_{i(t-1)(\omega)}} V_{it\omega}(x|x_{it}(\omega)) = \left\{x \in \mathbb{Z} | -z_{i(t-1)}(\omega) \leqslant x \leqslant x_{it(\omega)}\right\}$;

(3) If $WTP^1_{it}(\omega) < p_t(\omega) < WTA^1_{it}(\omega)$, then $x_{it}(\omega) \in \mathbb{Z} \bigcup \left[-z_{i(t-1)}(\omega), +\infty\right]$. In fact, for any given $x_{it}(\omega) \in \mathbb{Z}$, $argmax_{x \in \mathbb{Z}, x \geqslant -z_{i(t-1)(\omega)}} V_{it\omega}(x|x_{it}(\omega)) = \left\{x_{it(\omega)}\right\}$;

(4) If $p_t(\omega) = WTP^1_{it}(\omega)$, then $x_{it}(\omega) \in \mathbb{Z} \bigcup \left[-z_{i(t-1)}(\omega), +\infty\right]$. In fact, for any given $x_{it}(\omega) \in \mathbb{Z}$, $argmax_{x \in \mathbb{Z}, x \geqslant -z_{i(t-1)(\omega)}} V_{it\omega}(x|x_{it}(\omega)) = \left\{x \in \mathbb{Z} | x \geqslant x_{it(\omega)}\right\}$;

(5) If $p_t(\omega) < WTP^1_{it}(\omega)$, then $x_{it}(\omega) = +\infty$, since for any given $x_{it}(\omega) \in \mathbb{Z}$, $argmax_{x \in \mathbb{Z}, x \geqslant -z_{i(t-1)(\omega)}} V_{it\omega}(x|x_{it}(\omega)) = +\infty$.

To compare the results in our generalized framework to the classical framework, we provide two remarks. First, when $\eta = 0$, our model is reduced to the classical (reference-independent) setting, and we have $WTA^1_{it}(\omega) = WTP^1_{it}(\omega)$ and cases (2)-(4) merge into a single case where the agent's valuation of the asset is equal to the prevailing market price and she is indifferent among any feasible positions. Second, when $\eta > 0$ and $lambda = 1$, that is a reference-dependent framework with no loss aversion, and we also have $WTA^1_{it}(\omega) = WTP^1_{it}(\omega)$. This implies that in order to make our bubbles robust the assumption of loss aversion is necessary. Thereafter, when we mention reference dependence, we always refer to reference dependence with loss aversion.

4.5 Existence of *REE*

In our previous analyses, we have taken the existence of *REE* as given. With the trading strategy of a representative agent (specified in the previous section), we can show that a *REE* must exist in the binary exchange economy. Such a result is stated in the following proposition.[21]

Proposition 5 *In any binary economy, given the initial information structure $\left\{S_{it}\right\}^{i=1,2\cdots I}_{t=1,2\cdots T}$, there exists a REE.*

[20] It is easy to show that with the standard conditions $\eta > 0, \lambda > 1, 0 < q_{it\omega} < 1$, we have $p^{2*}_{i(t+1)\omega} < WTP^1_{it}(\omega) < WTA^1_{it}(\omega) < p^{1*}_{i(t+1)\omega}$.

[21] The existence result of *REE* in any binary economy can be easily extended to economies with more general settings since the whole analysis can be decomposed into many sub-cases, every one of which will follow the logic of analysis in a binary economy.

5 Examples and Comparisons

For ease of illustration, without loss of intuition, we focus on the simplest symmetric setting to see how a strong bubble can appear and sustain in a REE, robust to small perturbations in parameters. Following Zheng (2014) we assume that there are three periods ($T = 3$), two types of agents ($I = 2$) and eight states ($N = |\Omega| = 8$, denoted as $\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8\}$) [22] where $d(\omega|\omega \in \{\omega_1, \omega_4\}) = d^*$, $d(\omega|\omega \in \Omega \backslash \{\omega_1, \omega_4\}) = 0$ and the priors are specified in Table 1.

Table 1: Heterogeneous Priors of Two Agents

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
π_A	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
π_B	a_4	a_5	a_6	a_1	a_2	a_3	a_7	a_8

To generate a bubble, the (price-and-trade-refined) information partitions across all periods are the following, the same as in Zheng (2014), which can be shown to be the unique candidate for the current setup (Lien, Zhang and Zheng, 2015).

$$S_{A1} = S_{A1}^{PX} = \{\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_8\}, \{\omega_6, \omega_7\}\}$$

$$S_{B1} = S_{B1}^{PX} = \{\{\omega_1, \omega_2, \omega_4, \omega_5, \omega_6, \omega_8\}, \{\omega_3, \omega_7\}\}$$

$$S_{A2}^{PX} = \{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5\}, \{\omega_6\}, \{\omega_7\}, \{\omega_8\}\}$$

$$S_{B2}^{PX} = \{\{\omega_4, \omega_5, \omega_6\}, \{\omega_1, \omega_2\}, \{\omega_3\}, \{\omega_7\}, \{\omega_8\}\}$$

$$S_{A3}^{PX} = S_{B3}^{PX} = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}, \{\omega_5\}, \{\omega_6\}, \{\omega_7\}, \{\omega_8\}\}$$

To get a better idea of how the above information structure works together with the belief structure and the dividend structure to generate a strong bubble, we divide the 8 states into 5 types. State ω_7 is called the bubble state because in period 1 at ω_7 both agents will find the asset worthless. ω_3 and ω_6 are so-called "semi-bubble " states where in period 1 only one agent knows for sure that the asset is worthless and he or she rationally waits for bigger gains instead of selling the asset immediately. ω_1 and ω_4 are dividend-generating states that provide positive return at the end of the third period. ω_2 and ω_5 are the so-called "dividend-affiliate" states that are necessary to keep consistence in trading patterns of agents within information subsets in the second period, and ω_8 is

[22] In order for a rational bubble to exist in a REE, at least 3 periods are needed, as shown in AMP (1993). It can also be shown that a bubble can exist in a 3-period 5-state economy with asymmetric settings, where one agent has informational superiority over the other. However, if we focus on symmetric setups where no agent is ex ante favored, 8 states are necessary to generate a bubble. For detailed discussion of the necessary conditions to support a rational bubble, see Zheng (2014) and Lien, Zhang and Zheng (2015).

the dummy state that creates an additional degree of freedom to support the equilibrium condition in the first period. In our symmetric setup for two agents, all of the above states are necessary and hence we need at least 8 states to generate bubbles. For further interpretation about these states, please refer to Lien, Zhang and Zheng (2015). It is also easy to see that with more periods and more states, the argument in this section is still valid as we can always reduce the more general setting to the simplest one by merging states with equal priors and assuming complete information in the third period.

We further simplify our model by assuming that both agents have the same endowments of risky assets ($e_1 = e_2 = e$). For ease of notations, let $q_1 = \frac{a_1}{a_1+a_2+a_3}$, $q_2 = \frac{a_4}{a_4+a_5}$, $q_3 = \frac{a_1+a_2+a_3+a_4+a_5}{a_1+a_2+a_3+a_4+a_5+a_8}$ and $q_4 = \frac{a_6}{a_6+a_7}$.

5.1 A Numerical Example

For illustrative purposes, we begin with a numerical example with the following priors:

Table 2: Heterogeneous Priors of Two Agents: A Numerical Example

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
π_A	$\frac{2}{9}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{4}{9}$
π_B	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{2}{9}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{4}{9}$

Also, set $d^* = 12$, $\lambda = 2$ and $\eta = 1$, which implies that the agent values the consumption utility the same as the gain-loss utility but values the disutility from loss twice as much as the utility from gain. By definition, we have $q_1 = \frac{2}{3} > q_2 = \frac{1}{2}$, $q_3 = q_4 = \frac{1}{2}$.

Assuming that the information structure is given as above, we can easily compute that $p_3(\omega) = d(\omega)$, for all $\omega \in \Omega$ and $p_2(\omega) = 8$ for $\omega \notin \{\omega_7, \omega_8\}$, $p_2(\omega) = 0$ for $\omega \in \{\omega_7, \omega_8\}$. With classical preferences, we have $p_1(\omega) = 4$ for all $\omega \in \Omega$. In contrast, with reference-dependent preferences, prices are not observable in period 1, but the WTP is 3.2 and the WTA is 4. See Figures 1, 2 and 3 for graphical illustrations of the dynamics of prices. The eight states can be classified into three subsets according to prices: $S_1 = \{\omega_1, \omega_4\}$, $S_2 = \{\omega_2, \omega_3, \omega_5, \omega_6\}$, $S_3 = \{\omega_7, \omega_8\}$, where S_1 consists of the dividend-generating states, S_2 consists of the semi-bubble states and the dividend-affiliate states, and S_3 consists of the bubble state and the dummy state. Figure 1 shows that the price of the asset in the dividend-generating states are strictly increasing over time, while Figure 3 shows that the price of the asset in the bubble state and the dummy state are decreasing over time. Though the price dynamic patterns are the same for the bubble state and the dummy state, these two states differ in the nature of mutual knowledge between the two agents about the overpricing in period 1. Figure 2 shows that the price of the asset in other states is first increasing and then decreasing, which provide the more informed agent an opportunity to sell the worthless asset to the less informed "greater fool".

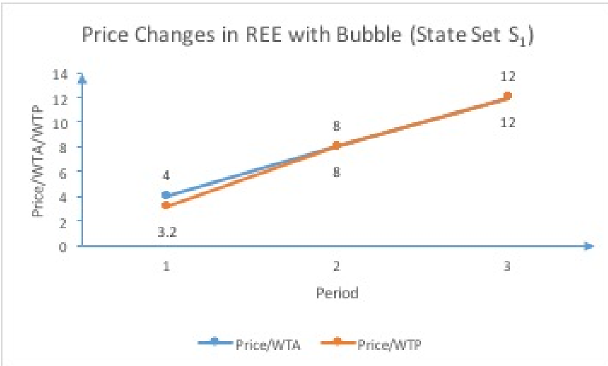

Figure 1: Price Dynamics (State Set S_1)

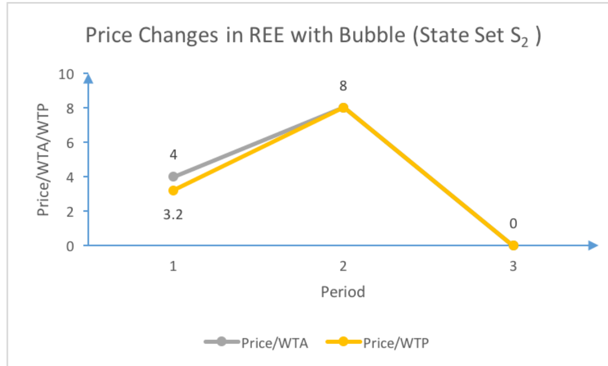

Figure 2: Price Dynamics (State Set S_2)

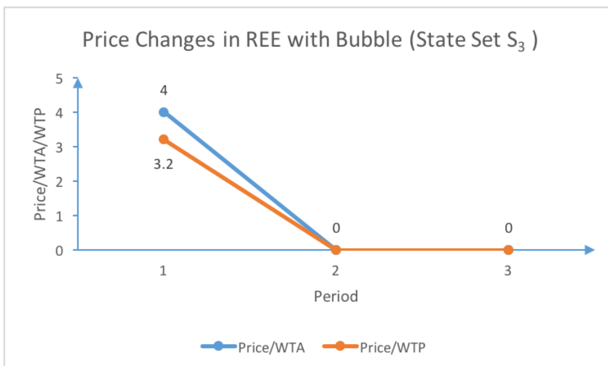

Figure 3: Price Dynamics (State Set S_3)

Taking the prices as given, one profile of trading strategies in the intersection of best response correspondences is that trades only happen in period 2 when player A sells all assets to player B in states $\{\omega_1, \omega_2, \omega_3\}$ while player B sells all assets to player A in states $\{\omega_4, \omega_5, \omega_6\}$. This trading profile is consistent with Table 3 in Section 5.2 and the prices (and WTA and WTP as well) are also consistent with Tables 4 and 5, thus they together constitute a REE for two cases (classical preferences and reference dependent preferences) respectively. Notice that both players know for sure that when ω_7 happens, the dividend must be 0, but the equilibrium price or the WTP /WTA are all strictly positive in period 1. Thus, we say a strong bubble occurs in period 1 at state ω_7 under both classical preferences and reference dependent preferences.

To see how a parameter change affect the robustness of the bubble, suppose that the dividend in ω_1 now increases from 12 to 15, while all the other parameters remain unchanged. For simplicity, we focus on the equilibrium under the original information structure. With classical preferences, $p_2(\omega) = 10$ for $\omega \in \{\omega_1, \omega_2, \omega_3\}$ and $p_2(\omega) = 8$ for $\omega \in \{\omega_4, \omega_5, \omega_6\}$ and $p_1(\omega_7) = max\{5,4\} = 5$,

$p_1(\omega_1) = p_1(\omega_6) = max\{4.75, 4.25\} = 4.75 < 5$ (from player B's perspective), which means that player A should be able to distinguish ω_6 and ω_7 from the price difference, contradicting with $S_{A1}^{PX} = \{\omega_6, \omega_7\}$. Thus, the bubble is no longer an equilibrium for the given information structure.

For the model with reference dependence, similarly, the key is to check whether the information partitions in the first period are consistent with price refinements via rational expectations. Suppose $e = 0.5$. For $\omega \in s_{A1}^{PX}(\omega_1)$, $WTA_{A1}(\omega) = 4.6$, $WTP_{A1}(\omega) = 3.95$; for $\omega \in s_{A1}^{PX}(\omega_7)$, $WTA_{A1}(\omega) = 4$, $WTP_{A1}(\omega) = 3.2$. Symmetrically, for $\omega \in s_{B1}^{PX}(\omega_1)$, $WTA_{B1}(\omega) = 4.1$, $WTP_{B1}(\omega) = 3.55$; for $\omega \in s_{B1}^{PX}(\omega_7)$, $WTA_{B1}(\omega) = 5$, $WTP_{B1}(\omega) = 4$. Notice that in every state, each player's WTA is still weakly higher than the other's WTP, and thus the market remains silent and prices are still unobservable, which successfully supports the original information structure. Then it follows that the existence of the bubble remains valid.

5.2 General Parameterized Model

5.2.1 Bubble Equilibrium under the Classical Framework and the Reference-Dependent Framework

Firstly, we compare the bubble equilibrium under the classical framework with the bubble equilibrium under the reference-dependent framework, and then check the robustness of bubbles under each framework. Consider the classical framework where $\eta = 0$ and $\mu(x) = 0$, then the definition of WTP requires that

$$\sum \pi_i(\omega')(p_{t+1}(\omega') - WTP_{it}^k(\omega)) = 0,$$

which implies that $WTP_{it}^k(\omega) = \sum_{\omega \in s_{it}^{PX}(\omega)} \pi_i(\omega') p_{t+1}(\omega') = E_{it}(p_{t+1}(\omega | s_{it}^{PX}(\omega)))$. Similarly, we also have $WTA_{it}^k(\omega) = E_{it}(p_{t+1}(\omega | s_{it}^{PX}(\omega)))$. That is, every agent will value the risky asset by their expected payoff in the next period, irrelevant to their current or expected position. According to the previous description of trading strategy, we must have $p_t(\omega) = max_i E_{it}(p_{t+1}(\omega | s_{it}^{PX}(\omega)))$. Hence, under the condition (i) $q_1 > q_2$ and (ii) $q_3 = q_4$, there is a REE $(X, P) \in \mathbb{Z}^{INT} \times (\mathbb{R}_+ \cup \{\emptyset\})^{NT}$ where there exist a bubble in period 1 and state ω_7, with the net trade profile specified in Table 3 and the price profile specified in Table 4.

Table 3: Profile of Net trades in REE: X

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
$t=1$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
$t=2$	$(e,-e)$	$(e,-e)$	$(e,-e)$	$(-e,e)$	$(-e,e)$	$(-e,e)$	$(0,0)$	$(0,0)$
$t=3$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$

Table 4: Profile of Prices in *REE*: P (Classical Framework)

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
$t=1$	$q_1q_3d^*$	$q_1q_3d^*$	$q_1q_3d^*$	$q_1q_3d^*$	$q_1q_3d^*$	$q_1q_3d^*$	$q_1q_3d^*$	$q_1q_3d^*$
$t=2$	q_1d^*	q_1d^*	q_1d^*	q_1d^*	q_1d^*	q_1d^*	0	0
$t=3$	d^*	0	0	d^*	0	0	0	0

It can be easily verified that the above (X,P) consists a *REE* in the specified pure exchange economy. As in period 1 both agents know that the dividend is 0 when ω_7 happens, and $p_1(\omega_7) = q_1q_3d^* > 0$, there is a first-order strong bubble in period 1 at state ω_7.

By the similarity in trading strategies, the above *REE* under the classic framework can evolve into a *REE* in the reference-dependent model with a slight adjustment in price profile under the conditions (i) and (ii), as is specified in Table 5.

Table 5: Profile of Prices in *REE*: P (Reference-Dependent Framework)

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
$t=1$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$t=2$	q_1d^*	q_1d^*	q_1d^*	q_1d^*	q_1d^*	q_1d^*	0	0
$t=3$	d^*	0	0	d^*	0	0	0	0

Notice that in period 2, prices are all set to be $max_i WTA^1_{i2}(\omega)$. For the set $\{\omega_1,\omega_2,\omega_3\}$, $WTA^1_{B2}(\omega) < p_2(\omega)$ and thus agent B has strictly positive incentive to sell all the risky asset he holds to agent A, while for the set $\{\omega_4,\omega_5,\omega_6\}$ the situation is reversed. As for period 1, notice that $WTA^1_{i1}(\omega) = q_1q_3d^* = q_1q_4d^* > \frac{(1+\eta)q_1q_3d^*}{(1+\eta)q_3+(1+\eta\lambda)(1-q_3)} = \frac{(1+\eta)q_1q_4d^*}{(1+\eta)q_4+(1+\eta\lambda)(1-q_4)} = WTP^1_{i1}(\omega), \forall i = A,B$, and thus the price is unobservable. It is worth mentioning that here the inequality $WTA^1_{i1}(\omega) > WTP^1_{i1}(\omega)$ can still hold even when the condition (ii) is violated.

5.2.2 Non-Robustness of the Classical Bubble

Zheng (2014) has shown that under certain types of perturbations, a rational bubble can persist, but robustness fails to hold in the general sense. Here we replicate the exercise in Zheng (2014) for the general perturbations, as a comparison to our robustness result in the next subsection.

Intuitively, with classical preferences, the prices in each state in period 1 are well defined even though there is no trade. Then to support a rational bubble, the given information structure requires that prices should not reveal any information about the true state, otherwise rational expectations imply that agents have finer information sets. Now, if the dividend in state ω_1 becomes slightly higher, the equilibrium price in ω_6 should be lower than that in ω_3 in period 2, which further implies that in period 1 player B is willing to bid (ask) higher than player A in state ω_7 and thus raises the market price. A similar argument can show that the period-1 price in state ω_1 also increases. However, the latter increase is relatively smaller, because for player B in period 1 the probability of reaching the state with higher dividend is smaller in state ω_6 than that in state ω_7. Thus, player A can also distinguish between these two states, which contradicts with the proposed information structure.

Now for formal proof, let $d'(\omega_1) = d^* + \varepsilon$ where $\varepsilon > 0$ and ε is sufficiently small, and other parameters remain unchanged. Suppose by contradiction that the previous information structure can still support a bubble. Then in period 2, under conditions (i) and (ii), the prices are

$$p_2(\omega) = \begin{cases} \frac{a_1(d^*+\varepsilon)}{a_1+a_2+a_3}(\equiv p_2^{1*}) & \omega \in \{\omega_1, \omega_2, \omega_3\} \\ \frac{a_1 d^*}{a_1+a_2+a_3}(\equiv p_2^{2*}) & \omega \in \{\omega_4, \omega_5, \omega_6\} \\ 0 & \omega \in \{\omega_7, \omega_8\} \end{cases} \tag{2}$$

For period 1, the analysis is similar and $p_1(\omega_7) = max\{E_{A1}(p_2(\omega_7), E_{B1}(p_2(\omega_7)\} = \frac{p_2^{1*}a_6}{a_6+a_7} = q_3 p_2^{1*}$, since $p_2^{1*} > p_2^{2*}$. Notice that the information partition has been refined by net trades and prices in period 1, and $\omega_3 \in s_{B1}^{PX}(\omega_7)$, $\omega_1 \in s_{A1}^{PX}(\omega_3)$, thus $p_1(\omega_7) = p_1(\omega_3) = p_1(\omega_1)$. However, $p_1(\omega_1) = max\{E_{A1}(p_2(\omega_1), E_{B1}(p_2(\omega_1)\} = max\{\frac{p_2^{1*}(a_1+a_2+a_3)+p_2^{2*}(a_4+a_5)}{a_1+a_2+a_3+a_4+a_5+a_8}, \frac{p_2^{2*}(a_1+a_2+a_3)+p_2^{1*}(a_4+a_5)}{a_1+a_2+a_3+a_4+a_5+a_8}\}$ $< \frac{p_2^{1*}(a_1+a_2+a_3+a_4+a_5)}{a_1+a_2+a_3+a_4+a_5+a_8} = q_3 p_2^{1*} = p_1(\omega_7)$, which contradicts with the equality $p_1(\omega_7) = p_1(\omega_1)$! That is, agent A will be able to distinguish ω_3 from ω_1 by observing the prices in different states, and the previous information structure will be updated to a different one. As has been shown in Lien, Zhang and Zheng (2015), the information structure specified in this example is the unique information structure under the simplest symmetric setting to support a strong bubble equilibrium, thus the classical rational bubbles are not robust to small changes in parameters.

5.2.3 Robustness of Reference-Dependent Bubbles

Now consider general arbitrary vibrations in both dividends and priors. Specifically, set $d'(\omega_j) = d(\omega_j) + \varepsilon_j, \forall j = 1, 2, \cdots, 8$ where ε_j is infinitesimal.[23] The new priors are specified below in Table 6, where $a'_{ij} = a_j + \sigma_{ij}$, σ_{ij} is sufficiently small and $\sum_j \sigma_{ij} = 0$, $i = A, B$, $j = 1, 2, \cdots, 8$.

Notice that the key to the failure of the classical model in the robustness issue is the sensitivity of prices to perturbations in parameters in the period when bubbles originally occur, which reveals additional information about the underlying states that is inconsistent with the original bubble-generating information structure. More intuitively, it lies in the fact that prices are always well-

[23]The change in dividend structure is common knowledge and dose not induce any private information.

Table 6: Heterogeneous Priors of Two Agents (Under Perturbations)

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
π_A	a'_{A1}	a'_{A2}	a'_{A3}	a'_{A4}	a'_{A5}	a'_{A6}	a'_{A7}	a'_{A8}
π_B	a'_{B4}	a'_{B5}	a'_{B6}	a'_{B1}	a'_{B2}	a'_{B3}	a'_{B7}	a'_{B8}

defined. Now in the reference-dependent settings, utility functions are still continuous in priors and dividends and robustness holds naturally for inequality conditions (like (i)) in periods 2 and 3. As for period 1 with $x_1(\omega) = 0, \forall \omega \in \Omega$, the gap between WTP and WTA guarantees that there are strict positive losses of net trades robust to small perturbations, which is different from the boundary conditions requiring indifferent preferences in the classical model. Accordingly, we can expect the REE with reference-dependent bubbles to persist regardless of the mildly disturbed economic environments.

For simplicity, we stick to the setup of binary economy and introduce a notion of **quasi-binary economy** to account for the perturbed version of the binary economy, in the sense that all the strict sensation of losses (gains) in the corresponding binary economy still at least weakly hold after a perturbation; that is, the perturbation in parameters will not overturn gains into losses and vice versa. Since the sign of the gain-loss utility is still deterministic and unchanged, many conclusions in a binary economy will continue to hold in a quasi-binary economy.

Now we show that the original information structure can still support a REE with a reference-dependent strong bubble under the new profile of parameters.

In period 3, by **Proposition 1**, we know that everyone values the risky asset according to its actual dividend level and thus $p_3(\omega) = d'(\omega)$ and $x_3(\omega) = 0$, $\forall \omega \in \Omega$.

In period 2, by utility maximization each agent's trading strategy can be derived accordingly. Denote $\frac{a'_{A1}(d^*+\varepsilon_1)(1+\eta)+(a'_{A2}\varepsilon_2+a'_{A3}\varepsilon_3)(1+\eta\lambda)}{a'_{A1}(1+\eta)+(a'_{A2}+a'_{A3})(1+\eta\lambda)} + \frac{a'_{A2}a'_{A3}\eta(\lambda-1)(e+x)|\varepsilon_2-\varepsilon_3|}{(a'_{A1}(1+\eta)+(a'_{A2}+a'_{A3})(1+\eta\lambda))(a'_{A1}+a'_{A2}+a'_{A3})}$ by $\underline{p}_{A2}(\omega)$, and $\frac{a'_{A1}(d^*+\varepsilon_1)+a'_{A2}\varepsilon_2+a'_{A3}\varepsilon_3}{a'_{A1}+a'_{A2}+a'_{A3}} - \frac{a'_{A2}a'_{A3}\eta(\lambda-1)(e+x)|\varepsilon_2-\varepsilon_3|}{(a'_{A1}(1+\eta)+(a'_{A2}+a'_{A3})(1+\eta\lambda))(a'_{A1}+a'_{A2}+a'_{A3})}$ by $\bar{p}_{A2}(\omega)$. In period 2, agents' trading strategies are characterized by **Claim 3**.

Claim 3 (Trading Strategy in a Quasi-Binary Economy) *Under the given prior structure, information structure and dividend structure, agent A's trading strategy is as follows:*

(1) For $\omega \in \{\omega_1, \omega_2, \omega_3\}$, $x \in argmax_{z\in\mathbb{Z}} V_{it\omega}(z|x)$ if and only if $\underline{p}_{A2}(\omega) \leqslant p_2(\omega) \leqslant \underline{p}_{A2}(\omega)$;

*(2) For $\omega \in \{\omega_4, \omega_5\}$, the case reduces to **Claim 2**;*

*(3) For $\omega \in \{\omega_6, \omega_7, \omega_8\}$, the case reduces to **Proposition 1**.*

By symmetric, agent B's trading strategy can be defined similarly with $\underline{p}_{B2}(\omega)$ and $\bar{p}_{B2}(\omega)$.

Consider the following trading profile in period 2, which is exactly the same as the period-2 trading profile in Table 2.

Table 7: Profile of Net trades in period 2: X (Under Perturbations)

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
$t=2$	$(e,-e)$	$(e,-e)$	$(e,-e)$	$(-e,e)$	$(-e,e)$	$(-e,e)$	$(0,0)$	$(0,0)$

The above trading profile can be supported if the following two conditions hold:

(S1) $WTA_{A2}(\omega|s_it(\omega_4)) = \frac{a'_{A4}(d^*+\varepsilon_4)+a'_{A5}\varepsilon_5}{a'_{A4}+a'_{A5}} < \bar{p}_{B2}(\omega)$ and $WTA_{A2}(\omega|s_it(\omega_6)) = \varepsilon_6 < \bar{p}_{B2}(\omega)$;

(S2) $WTA_{B2}(\omega|s_it(\omega_1)) = \frac{a'_{B4}(d^*+\varepsilon_1)+a'_{B5}\varepsilon_2}{a'_{B4}+a'_{B5}} < \bar{p}_{A2}(\omega)$ and $WTA_{A2}(\omega|s_it(\omega_3)) = \varepsilon_3 < \bar{p}_{A2}(\omega)$.

Notice that ε_j and $a'_{ij} - a_j = \sigma_{ij}$ are sufficiently small and those valuations are continuous, so we expect $WTA_{A2}(\omega|s_it(\omega_4))$ to be close to $\frac{a_4 d^*}{a_4+a_5} = q_2 d^*$ and $\bar{p}_{B2}(\omega)$ close to $\frac{a_1 d^*}{a_1+a_2+a_3} = q_1 d^*$. Since $q_1 > q_2$ by condition (i), it is easy to have $(S1)$ hold under small perturbations. A similar analysis implies that $(S2)$ will also hold.

If this is the case, the prices in period 2 are:

$$p_2(\omega) = \begin{cases} \frac{a'_{A1}(d^*+\varepsilon_1)+a'_{A2}\varepsilon_2+a'_{A3}\varepsilon_3}{a'_{A1}+a'_{A2}+a'_{A3}} - \frac{2a'_{A2}a'_{A3}\eta(\lambda-1)e|\varepsilon_2-\varepsilon_3|}{(a'_{A1}(1+\eta)+(a'_{A2}+a'_{A3})(1+\eta\lambda))(a'_{A1}+a'_{A2}+a'_{A3})}(\equiv p_2^{1*}) & \omega \in \{\omega_1,\omega_2,\omega_3\} \\ \frac{a'_{B1}(d^*+\varepsilon_4)+a'_{B2}\varepsilon_5+a'_{B3}\varepsilon_6}{a'_{B1}+a'_{B2}+a'_{B3}} - \frac{2a'_{B2}a'_{B3}\eta(\lambda-1)e|\varepsilon_4-\varepsilon_5|}{(a'_{B1}(1+\eta)+(a'_{B2}+a'_{B3})(1+\eta\lambda))(a'_{B1}+a'_{B2}+a'_{B3})}(\equiv p_2^{2*}) & \omega \in \{\omega_4,\omega_5,\omega_6\} \\ \varepsilon_7 & \omega \in \{\omega_7\} \\ \varepsilon_8 & \omega \in \{\omega_8\} \end{cases} \tag{3}$$

Remember that $p_2^{1*} \approx q_1 d^* \approx p_2^{2*}$ is a key feature to determine the signs of sensation in period 1.

In period 1, by the same logic we know that the silent market can persist in an *REE* when no one wants to pay more for a unit of risky asset than anyone's reservation price, that is

(S3) $min_i(WTA_{i1}(\omega)) \geqslant max_i(WTP_{i1}(\omega)), \forall \omega \in \Omega$.

Specifically, denote $q'_{i1} = a'_{i1} + a'_{i2} + a'_{i3}$, $q'_{i2} = a'_{i4} + a'_{i5}$ and $q'_{i3} = a'_{i8}$ where $i = A, B$ and then use agent A for illustration:[24]

(a) For $\omega \in s_{A1}^{PX}(\omega_1)$, $WTA_{A1}(\omega) = \frac{q'_{A1}p_2^{1*}+q'_{A2}p_2^{2*}+q'_{A3}\varepsilon_8}{q'_{A1}+q'_{A2}+q'_{A3}} - \frac{q'_{A1}q'_{A2}\eta(\lambda-1)e|p_2^{1*}-p_2^{2*}|}{q'_{A3}(1+\eta\lambda)+(q'_{A1}+q'_{A2})(1+\eta)}$,

$WTP_{A1}(\omega) = \frac{(q'_{A1}p_2^{1*}+q'_{A2}p_2^{2*})(1+\eta)+q'_{A3}\varepsilon_8(1+\eta\lambda)}{q'_{A3}(1+\eta\lambda)+(q'_{A1}+q'_{A2})(1+\eta)} + \frac{q'_{A1}q'_{A2}\eta(\lambda-1)e|p_2^{1*}-p_2^{2*}|}{q'_{A3}(1+\eta\lambda)+(q'_{A1}+q'_{A2})(1+\eta)}$.

[24] Again, the case for agent B is completely symmetric.

(b) For $\omega \in s^{PX}_{A1}(\omega_7)$, $WTA_{A1}(\omega) = \frac{a'_{A6}p_2^{2*}+a'_{A7}\varepsilon_7}{a'_{A6}+a'_{A7}}$,
$WTP_{A1}(\omega) = \frac{a'_{A6}p_2^{2*}(1+\eta)+a'_{A7}\varepsilon_7(1+\eta\lambda)}{a'_{A6}(1+\eta)+a'_{A7}(1+\eta\lambda)}$.

We then need to check the plausibility of $(S3)$. Note that the continuity of valuations implies that $WTP_{A1}(\omega|s^{PX}_{i1}(\omega_1))$ and $WTP_{i1}(\omega|s^{PX}_{A1}(\omega_7))$ should be close to $\frac{(1+\eta)q_1q_3d^*}{(1+\eta)q_3+(1+\eta\lambda)(1-q_3)} = \frac{(1+\eta)q_1q_4d^*}{(1+\eta)q_4+(1+\eta\lambda)(1-q_4)}$, while $WTA_{A1}(\omega|s^{PX}_{i1}(\omega_1))$ and $WTA_{i1}(\omega|s^{PX}_{A1}(\omega_7))$ should be close to $q_1q_3d^* = q_1q_4d^*$. When the perturbations are small enough, $(S3)$ will naturally be satisfied and the prices in period 1 are unobservable. Thus, the original profile of net trades can still be supported in a *REE* in the perturbed environment.

As for the existence of strong bubble, notice that at state ω_7 in period 1, both agents know for sure that the dividend is going to be ε_7, an infinitesimal amount of money, but are willing the evaluate the risky asset as high as $\frac{a'_{A6}p_2^{2*}(1+\eta)+a'_{A7}\varepsilon_7(1+\eta\lambda)}{a'_{A6}(1+\eta)+a'_{A7}(1+\eta\lambda)}$ (for A) and $\frac{a'_{B6}p_1^{2*}(1+\eta)+a'_{B7}\varepsilon_3(1+\eta\lambda)}{a'_{B6}(1+\eta)+a'_{B7}(1+\eta\lambda)}$ (for B), which are comparable to d^* in size and hence are clearly higher then the underlying fundamental value. Therefore, a robust strong bubble persists in equilibrium, regardless of the perturbations in parameters, as long as the perturbation is small enough.

5.3 The Limit of Bubble Size and Bubble Frequency

Lien, Zhang and Zheng (2015) shows that the size of strong bubbles can approach the highest dividend level in equilibrium, and it is also possible to have a strong bubble that appears almost for sure in equilibrium. As is shown below, under the reference-dependent framework, the limit of bubble size and bubble frequency can be also approached and the same results are preserved. We first consider the limit of bubble size. In the reference-dependent model, following literature, we can define the **relative bubble size** θ for the bubble in period t at state ω as $\theta \equiv \frac{min_i(WTP^1_{it}(\omega))}{max_\omega(d(\omega))}$. It is hence easy to see that in our example in the previous section the bubble size without perturbations is $\theta = \frac{(1+\eta)q_1q_3}{(1+\eta)q_3+(1+\eta\lambda)(1-q_3)}$, which is increasing in q_1 and q_3. When agent A thinks that the probabilities of states ω_2, ω_3, and ω_8 are sufficiently smaller than other states and (by symmetry) agent B thinks that the probabilities of states ω_4, ω_5, and ω_8 are sufficiently smaller than other states, q_1 and q_3 will both approach 1. Consider the example in Table 8, and we can see that as $\sigma \to 0$, the bubble size $\theta \to 1$ since $q_1, q_3 \to 1$.

Table 8: Heterogeneous Priors of Two Agents: An Example for Bubble Size

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
π_A	$\frac{1}{2}-3\sigma$	$\frac{1}{2}\sigma$	$\frac{1}{2}\sigma$	$\frac{1}{2}\sigma$	$\frac{1}{2}\sigma$	$\frac{1}{2}-\sigma$	σ	σ
π_B	$\frac{1}{2}\sigma$	$\frac{1}{2}\sigma$	$\frac{1}{2}-\sigma$	$\frac{1}{2}-3\sigma$	$\frac{1}{2}\sigma$	$\frac{1}{2}\sigma$	σ	σ

Note that in our framework the bubble size can never be greater than the highest dividend, which serves as the limit. In fact, when everyone possesses rational expectations, it can never be the case that anyone will buy a piece of asset with the price higher than the highest possible fundamental value or be reluctant to sell a piece with the price higher than the highest possible fundamental value. Our examples shows that as long as the bubble size is within the limit, by manipulating agents' priors, the bubble can be arbitrarily large.

Another question of our interest is to understand how likely a bubble can occur in equilibrium under the reference-dependent framework. Our analysis shows that by having appropriate settings on agents' priors of the states, a bubble can appear almost for sure in equilibrium. Take the priors specified in Table 9 as an example, and we have bubble frequency $p = 1 - 2\sigma - \sigma^2 \to 1$ as $\sigma \to 0$.

Table 9: Heterogeneous Priors of Two Agents: An Example for Bubble Frequency

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
π_A	$\frac{3}{7}\sigma$	$\frac{1}{7}\sigma$	$\frac{1}{7}\sigma$	$\frac{1}{7}\sigma$	$\frac{1}{7}\sigma$	σ	$1-2\sigma-\sigma^2$	σ^2
π_B	$\frac{1}{7}\sigma$	$\frac{1}{7}\sigma$	σ	$\frac{3}{7}\sigma$	$\frac{1}{7}\sigma$	$\frac{1}{7}\sigma$	$1-2\sigma-\sigma^2$	σ^2

5.4 Degree of Robustness

During the discussion in the previous section, we only use continuity property to obtain some intuition about why the bubble is robust to perturbations. In this subsection, we present a numerical example to investigate to what extent the perturbation in parameters can be to guarantee the existence of the bubble.

Consider the numerical example in section 5.1. The gap between WTP and WTA can be as large as 20% (see the expression for WTP and WTA in section 5.2.1 and the ratio of WTP over WTA is $\frac{(1+\eta)}{(1+\eta)q_3+(1+\eta\lambda)(1-q_3)} = \frac{4}{5}$), which is unlikely to occur only by diminishing sensitivity. For simplicity here we assume that the priors are fixed, dividends for most of the states are fixed, and only focus on the perturbations in dividend for state ω_1, represented by ε_1. By $(S1)-(S3)$, a simply calculation shows that the only condition needed to be satisfied to support the bubble equilibrium is that $-\frac{d^*}{5} \leqslant \varepsilon_1 \leqslant \frac{d^*}{4}$, which indicates a rather large range of the possible values of ε_1. We can easily show similar naturally-unbinding conclusions will hold for more complicated forms of perturbations. These exercises imply that the robustness is quite strong in the sense that the existence of bubble will persist for moderate (not just infinitesimal) perturbations.

5.5 Applicability of the Reference-Dependent Model

In the previous two subsections we show that the strong bubble can be quite common and robust, but both of the analyses are based on a given information structure. Admittedly, we have treated the formation of initial information partitions as exogenously given in this paper, and if the information structure is arbitrarily determined, the chance to have a strong bubble can be very small, which is a realistic reflection of the real world economy. Another shortcoming of our current model is that the setup is very abstract and may be over simplified, as we are looking at the simplest situation (say, the binary economy environment) which captures the essence of the problem.

After all, we expect the following insights to be feasible for the general setup even if some computational conclusions may fail to hold. First, for the occurrence of robust speculative bubbles, the story of "greater fool" still applies in the sense that everyone is willing to bid a (clearing) price of the risky asset which is much higher than its fundamental value, with the hope to sell the "overvalued" asset to a "less-informed" agent. While on the equilibrium path in some state an agent does successfully find such a "greater fool" and makes profit by selling the "over-valued" asset, in some other state (for example the bubble state ω_7 in our example) every agent ends up with holding the "over-valued" asset at hand when the bubble crashes. The chance of successful speculation and the chance of failure of escaping balances in a way such that agents ex ante are willing to hold the asset even though they know it is "over-valued". Second, the intuition behind the robustness feature of the reference-dependent framework is that the assumption of risk aversion helps generate a gap between WTA and WTP and hence convert the equilibrium conditions from boundary conditions to interior ones, which easily hold under small perturbations in parameters due to the property of continuity.

Although we are confident that our model has done a good job in explaining the surprising phenomenon of strong bubbles and making bubbles robust to parameters changes in environment, we are not optimistic about whether our model can provide precise predictions about the pricing dynamics and occurrence of bubbles in the real world. The primary goal of this paper is to theoretically show the possibility of coexistence of rational expectations and speculative bubbles in a robust way. It is true that agents in an asset market are not completely rational, and irrationality may induce bubbles in a more simple and elegant way. However, as far as we are concerned, permanent irrationality seems to be so extreme as to ignore the fact that agents are learning over time and irrational agents may quit the market once they suffer from the loss. In contrast, our model does not rely on this kind of ignorance and agents are continuing to fuel a bubble rationally no matter how sophisticated or experienced they are.

6 Conclusion

This paper presents a model of robust speculative bubbles, to resolve the fragility issue of bubble equilibrium in the AMP framework. After replacing the classical utility with the reference-dependent loss-averse utility, speculative bubble can be robust to any form of small perturbations in the economic environments since the binding boundary conditions in the classical model are now converted into non-binding interior conditions. Indeed our model serves as an extension of the classical model by incorporating it as a special case where agents only care about the consumption utility, and it requires weaker conditions to support a REE with strong bubbles. It is also worth-mentioning that most features (other than the non-robustness drawback) for strong bubbles in the AMP framework are preserved in our model. In particular, we show that the size of the robust bubble can approach the level of the highest dividend when agents attach high enough priors to those high-dividend states and it is also possible that the robust bubble occurs almost for sure in certain REE.

In short, our work attempts to establish a new way to cope with the non-robustness issue of classical rational bubbles without diverting too far away from the classical framework. From a broader perspective, our work also contributes to research on improving classical models by incorporating more realistic behavioral features, with higher explanatory power and broader applicable scope. As for contribution in methodology, our model provides an analysis to a new setting in which the reference points are endogenously formed as agents' rational expectations in the asset market where prices are also endogenously determined, while in the previous models by Koszegi and Rabin (2006, 2007) prices are taken as exogenously given.

One possible extension of the current work is to show whether the model is robust to higher-order knowledge (Conlon, 2004). Notice that in our model, the definition of bubble is of first order in the sense that it is first-order mutual knowledge that the asset has been over-priced. A stronger definition of robust bubble may require that the over-pricing is kth-order mutual knowledge to agents, that is, everyone knows that everyone knows that $\cdots$ (k times) the risky asset has been over-priced. Intuitively, more states and periods and more complicated information structure will be needed to handle higher-order knowledge. Another potential direction for future study is to explicitly consider the formation of the initial information structure. In this paper we assume for simplicity that it is exogenously given, which is not a very realistic reflection of the real world. The dynamic interactions among agents trading an asset across different periods may serve as a source for the endogenized formation of the initial information structure.

A Appendix: General Results

A.1 Incentives for Assuming Binary Economy

As we have pointed out in Section 4.3, binary economy is assumed to simplify the tedious discussion. That is, each term can easily be classified as gain or loss. We use the proof of **Proposition 3** to give a detailed illustration.

Consider the binary prospect $(p^*, q; 0, 1-q)$ and $x_i \in \mathbb{N}_+$,[25]

$V_{it\omega}(x_i|0) = m_{it} + q(z_{i(t-1)} + x_i)p^* - x_i p_t(\omega) + q^2\mu(x_i(p^* - p_t(\omega))) + q(1-q)\mu((z_{i(t-1)} + x_i)p^* - x_i p_t(\omega)) + q(1-q)\mu(-z_{i(t-1)}p^* - x_i p_t(\omega)) + (1-q)^2\mu(-x_i p_t(\omega))$

$V_{it\omega}(0|0) = m_{it} + qz_{i(t-1)}p^* + q(1-q)\mu(z_{i(t-1)}p^*) + q(1-q)\mu(-z_{i(t-1)}p^*)$

Claim 4 *In any REE, for any t and any ω, $(o \leqslant) p_t(\omega) \leqslant p^*$.*

With the claim, the expression is now deterministic and the following are equivalent:

$$V_{it\omega}(x_i|0) - V(0|0) = x_i(qp^* - p_t(\omega)) + x_i q\eta(p^* - p_t(\omega)) - x_i(1-q)\eta\lambda p_t(\omega) \geqslant 0 \quad (4)$$

$$V_{it\omega}(1|0) - V(0|0) = qp^* - p_t(\omega) + q\eta(p^* - p_t(\omega)) - (1-q)\eta\lambda p_t(\omega) \geqslant 0 \quad (5)$$

This is the main result of **Proposition 3**: $V_{it\omega}(x_i|0) - V(0|0) \geqslant 0 \Longleftrightarrow V_{it\omega}(1|0) - V(0|0) \geqslant 0$.

Although the assumption of binary economy provides us with the essential intuition of reference-dependent bubbles, it may be unrealistic in the real world. We have claimed without clarification that many conclusions will hold for general prospects despite for the tiresome enumeration of possible regions of prices. Intuitively, the main difference lies between the binary case and the ternary case since that is where deterministic analysis fails. As for more complicated prospects, induction can be used to show that nothing essential changes.

A.2 General Economy: Proposition 3'

Proposition 3' *Given t andω, suppose agent i faces some prospect, $\forall i = 1, 2 \cdots I$. With status quo as the reference point, if agent i is willing to buy or sell some units of the risky asset, then she must be willing to buy or sell one unit. Furthermore, in any feasible REE, the price $p_t(\omega)$ is unobservable if and only if $WTP^1_{it}(\omega) < WTA^1_{jt}(\omega)$, $\forall i \neq j$.*

Intuitively, the proposition says that under a certain expected position, if an agent is willing to deviate from the expectation by x units , then she must be willing to deviate by y units for all $y \leqslant x$. This resembles the idea of decreasing WTP or increasing WTA out of the assumption of diminishing sensitivity and second-order risk-aversion, and implies the similarity between loss aversion and risk-aversion. However, it is somehow difficult to directly compare different WTAs/WTPs due

[25] For simplicity, we use m_{it} to stand for $m_{it}(\omega)$ and $z_{i(t-1)}$ to stand for $z_{i(t-1)}(\omega)$.

to the dynamics of reference point updating process.

Recall we use $WTP^1_{it}(\omega|x_i)$ ($WTA^1_{it}(\omega|x_i)$) to denote the willingness to pay (willingness to accept) of agent i at period t in state ω for an additional unit of risky asset with respect to the reference point x_i, and we simplify the notation by setting $WTP^1_{it}(\omega|0) = WTP^1_{it}(\omega)(WTA^1_{it}(\omega|0) = WTA^1_{it}(\omega)$. Then by **Proposition 3'**, a computationally equivalent condition of utility maximization can be drawn.

Corollary 3 *To determine whether $x_i \in \mathbb{N}_+$ is a REE trading decision for agent i at period t in state ω, it suffices to compare $WTP^1_{it}(\omega|x_i)$, $WTA^1_{it}(\omega|x_i)$ and $p_t(\omega)$. Specifically,*
$x_i \in argmax_{x\in\mathbb{Z}}\, V_{it\omega}(x|x_i),\;\; s.t.\, x \geqslant -z_{i(t-1)}(\omega) \Longleftrightarrow WTP^1_{it}(\omega|x_i) \leqslant p_t(\omega) \leqslant WTA^1_{it}(\omega|x_i).$

A.3 General Economy: Trading Strategy

In this section we will discuss whether **Propositions 4 and 5** can straightforwardly be generalized to the ternary case or more complicated economies. Unfortunately, some conclusions in the binary economy are too special to be general, but the main intuitions still apply.

A.3.1 Case of Purchasing: Endowment Effect of Risk

Claim 5 *If $V_{it\omega}(1|0) > V_{it\omega}(0|0)$, then for all $x_i \in \mathbb{N}_+$, $V_{it\omega}(x_i+1|x_i) > V_{it\omega}(x_i|x_i)$.*

The proof of this claim reveals the intuition for the so-called endowment effect of risk. Notice that the term $\frac{q_2q_3\eta(\lambda-1)z_{i(t-1)}p_2^*}{1+q_1\eta+(q_2+q_3)\eta\lambda}$ does not appear in the binary case and is positively correlated with the anticipated position which coincides with status quo here. In Koszegi and Rabin (2007), the endowment effect of risk implies that when expecting risk, one will be less risk averse for the additional risk. Intuitively, when an agent anticipates risk, and thus has stochastic reference point, she will go through mixed sensation and decreased utility (compared to deterministic reference point) due to loss aversion. Then after an additional risk prospect is added, there is a mixed sensation and a bad outcome is no longer a pure loss. Again by loss aversion, the mixed sensation of loss is not as strong as previous losses and thus the agent appears less risk-averse for the additional risk, which leads to higher willingness to pay.

One important implication of this claim is that it excludes a huge group of situations in terms of REE.

Claim 6 *Given state ω, period t and reference point as the status quo, if agent i strictly prefers purchasing a positive number of units of risky asset relative to the status quo, then there cannot be any REE where agent i buys or stays at status quo.*

Proof. Suppose by way of contradiction that $x_i \geqslant 0$ is the corresponding REE trading choice. From **Proposition 3'**, $\exists k \in \mathbb{N}_+, V_{it\omega}(k|0) > V_{it\omega}(0|0) \Longrightarrow V_{it\omega}(1|0) > V_{it\omega}(0|0) \Longrightarrow V_{it\omega}(x_i+1|x_i)) > V_{it\omega}(x_i|x_i)$, a contradiction with utility maximization. □

In the proof above, the first induction $V_{it\omega}(k|0) > V_{it\omega}(0|0) \Longrightarrow V_{it\omega}(1|0) > V_{it\omega}(0|0)$ holds since under the same reference point, the idea of diminishing marginal utility in reference-independent settings applies and one should be willing to buy one before one wants to buy more. As for the second one, notice that the reference point has changed and there is nothing to do with diminishing sensitivity as the distances from the reference points are the same. Instead, it is due to the endowment effect of risk discussed above.

A.3.2 Case of Selling: Insuring for Stochastic Gains

Following the same logic of analysis in the last section, we wanted to prove that $V_{it\omega}(-1|0) > V_{it\omega}(0|0) \Longrightarrow V_{it\omega}(-x_i-1|-x_i)) > V_{it\omega}(-x_i|-x_i)$, which, unfortunately, does not necessarily hold. For example, consider $p_t(\omega) < p_1^* + z_{i(t-1)}(p_2^* - p_1^*) < p_1^* + (z_{i(t-1)} - x_i)(p_2^* - p_1^*)$, then

$V_{it\omega}(-1|0) > V_{it\omega}(0|0) \Longleftrightarrow p_t(\omega) > (q_1 p_1^* + q_2 p_2^*) + \frac{q_1 q_2 \eta(1-\lambda) z_{i(t-1)}(p_1^* - p_2^*)}{1+(q_1+q_2)\eta\lambda + q_3\eta}$

$V_{it\omega}(-1-x_i|-x_i) > V_{it\omega}(-x_i|-x_i) \Longleftrightarrow p_t(\omega) > (q_1 p_1^* + q_2 p_2^*) + \frac{q_1 q_2 \eta(1-\lambda)(z_{i(t-1)} - x_i)(p_1^* - p_2^*)}{1+(q_1+q_2)\eta\lambda + q_3\eta}$

Then we have the reverse conclusion: $V_{it\omega}(-x_i-1|-x_i)) > V_{it\omega}(-x_i|-x_i) \Longrightarrow V_{it\omega}(-1|0) > V_{it\omega}(0|0)$. Actually, $WTA_{it}^1(\omega|-k)$ is increasing with k for $\forall k \in \mathbb{N}_+$ and $0 \leqslant k \leqslant z_{i(t+1)} - 1$. Computationally, the term negatively related to the expected position comes from the fact that the price is so low as to transfer a gain into a loss compared to the reference point valuation (that is $B1 \leqslant 0$, but for the corresponding term in $V_{it\omega}(-x_i|-x_i)$, $B1 = (z_{i(t-1)} - x_i)(p_1^* - p_2^*) \geqslant 0$). Furthermore, the induction of loss aversion parameter $\lambda > 1$ helps avoid the cancelling out of the term $(z_{i(t-1)} - x_i)(p_1^* - p_2^*)$, which can be seen as an extra gain (compared to the case without the term) positively correlated with anticipated position and thus lower the level of WTA.

Intuitively, consider the case as "insuring" for stochastic gains, that is, the agent is choosing the smallest certain compensation for giving up an additional gamble based on the reference point involving that gamble. Note that an additional payment cannot create stronger gain-loss sensation than an additional stochastic gamble due to loss aversion, and the more the underlying risk is, the larger the gap is. That is, certain payment relatively increases gain-loss utility and thus decreases required WTP. Thus, the agent is willing to sell the risky asset at a price lower than its expected value and the willingness to accept increases with the amount having been sold. In other words, we show that even without increasing marginal cost, the seller will still be more reluctant to sell an additional unit when having sold many.

Although the strong conclusions for binary case fail to hold any more, it can be seen that the present one is more realistic. After all, the binary economy "radically" predicts that when an agent is strictly willing to sell, she will sell everything.

Before referring to the trading strategy, we firstly show some mathematical results.

(1) If $p_t(\omega) < p_1^* + (z_{i(t-1)} - x_i)(p_2^* - p_1^*)$, then

$$V(-x_i-1|-x_i) > V(-x_i|-x_i) \Longleftrightarrow p_t(\omega) > (q_1p_1^*+q_2p_2^*) + \frac{q_1q_2\eta(1-\lambda)(z_{i(t-1)}-x_i)(p_1^*-p_2^*)}{1+(q_1+q_2)\eta\lambda+q_3\eta}.$$

(2) If $p_t(\omega) > (z_{i(t-1)}-x_i)p_2^*$, then

$$V(-x_i-1|-x_i) > V(-x_i|-x_i) \Longleftrightarrow p_t(\omega) > (q_1p_1^*+q_2p_2^*) + \frac{q_2q_3\eta(1-\lambda)(z_{i(t-1)}-x_i)p_2^*}{1+q_1\eta\lambda+(q_2+q_3)\eta}.$$

(3) If $p_1^*+(z_{i(t-1)}-x_i)(p_2^*-p_1^*) \leqslant p_t(\omega) \leqslant p_2^*$, then

$$V(-x_i-1|-x_i) > V(-x_i|-x_i) \Longleftrightarrow p_t(\omega) > \frac{q_1p_1^*(1+q_1\eta\lambda+(q_2+q_3)\eta)+q_2p_2^*(1+(q_1+q_2)\eta\lambda+q_3\eta)}{1+q_1\eta\lambda+q_3\eta+q_2(q_2+q_3)\eta\lambda+q_1q_2\eta}.$$

(4) If $p_2^* \leqslant p_t(\omega) \leqslant p_2^*(z_{i(t-1)}-x_i)$, then

$$V(-x_i-1|-x_i) > V(-x_i|-x_i) \Longleftrightarrow p_t(\omega) > \frac{(q_1p_1^*+q_2p_2^*)(1+q_1\eta\lambda+(q_2+q_3)\eta)}{1+q_1\eta\lambda+(q_3+q_2)\eta+q_2q_3\eta(\lambda-1)}.$$

With **Proposition 3'** and the discussion above, we can directly infer the trading strategy as a function of prices.

(1) If $p_t(\omega) < p_2^*+z_{i(t-1)}(p_2^*-p_1^*)$, then $\forall x_i \in \mathbb{Z}\bigcup\left[-z_{i(t-1)}(\omega),+\infty\right]$,

$$x_i \in argmax_{x\in\mathbb{Z}}\, V_{it\omega}(x|x_i),\ \ s.t.\ x \geqslant -z_{i(t-1)(\omega)} \Longleftrightarrow$$
$$\frac{(q_1p_1^*+q_2p_2^*)(1+\eta)+q_1q_2\eta(\lambda-1)(z_{i(t-1)}+x_i)(p_1^*-p_2^*)}{1+(q_1+q_2)\eta+q_3\eta\lambda} \leqslant p_t(\omega) \leqslant (q_1p_1^*+q_2p_2^*) + \frac{q_1q_2\eta(1-\lambda)(z_{i(t-1)}+x_i)(p_1^*-p_2^*)}{1+(q_1+q_2)\eta\lambda+q_3\eta}.$$

(2) If $p_2^*+z_{i(t-1)}(p_2^*-p_1^*) \leqslant p_t(\omega) < p_1^*+z_{i(t-1)}(p_2^*-p_1^*)$, then $\forall x_i \in \mathbb{Z}\bigcup\left[-z_{i(t-1)}(\omega),+\infty\right]$,

$$x_i \in argmax_{x\in\mathbb{Z}}\, V_{it\omega}(x|x_i),\ \ s.t.\ x \geqslant -z_{i(t-1)(\omega)} \Longleftrightarrow$$
$$\frac{q_1p_1^*(1+\eta)+q_2p_2^*(1+q_1\eta\lambda+(1-q_1)\eta)}{1+q_1\eta+q_2(q_2+q_3)\eta+q_3\eta\lambda+q_1q_2\eta\lambda} \leqslant p_t(\omega) \leqslant (q_1p_1^*+q_2p_2^*) + \frac{q_1q_2\eta(1-\lambda)(z_{i(t-1)}+x_i)(p_1^*-p_2^*)}{1+(q_1+q_2)\eta\lambda+q_3\eta}.$$

(3) If $p_1^*+z_{i(t-1)}(p_2^*-p_1^*) \leqslant p_t(\omega) < p_2^*$, then $\forall x_i \in \mathbb{Z}\bigcup\left[-z_{i(t-1)}(\omega),+\infty\right]$,

$$x_i \in argmax_{x\in\mathbb{Z}}\, V_{it\omega}(x|x_i),\ \ s.t.\ x \geqslant -z_{i(t-1)(\omega)} \Longleftrightarrow$$
$$\frac{q_1p_1^*(1+\eta)+q_2p_2^*(1+q_1\eta\lambda+(1-q_1)\eta)}{1+q_1\eta+q_2(q_2+q_3)\eta+q_3\eta\lambda+q_1q_2\eta\lambda} \leqslant p_t(\omega) \leqslant \frac{q_1p_1^*(1+q_1\eta\lambda+(q_2+q_3)\eta)+q_2p_2^*(1+(q_1+q_2)\eta\lambda+q_3\eta)}{1+q_1\eta\lambda+q_3\eta+q_2(q_2+q_3)\eta\lambda+q_1q_2\eta}.$$

(4) If $p_2^* \leqslant p_t(\omega) < p_2^*z_{i(t-1)}$, then $\forall x_i \in \mathbb{Z}\bigcup\left[-z_{i(t-1)}(\omega),+\infty\right]$,

$$x_i \in argmax_{x\in\mathbb{Z}}\, V_{it\omega}(x|x_i),\ \ s.t.\ x \geqslant -z_{i(t-1)(\omega)} \Longleftrightarrow$$
$$\frac{q_1p_1^*(1+\eta)+q_2p_2^*(1+(1-q_3)\eta\lambda+q_3\eta)}{1+q_1\eta+q_2q_3\eta+q_3\eta\lambda+q_2(q_1+q_2)\eta\lambda} \leqslant p_t(\omega) \leqslant \frac{q_1p_1^*(1+q_1\eta\lambda+(q_2+q_3)\eta)+q_2p_2^*(1+(q_1+q_2)\eta\lambda+q_3\eta)}{1+q_1\eta\lambda+q_3\eta+q_2(q_2+q_3)\eta\lambda+q_1q_2\eta}.$$

(5) If $p_2^*z_{i(t-1)} \leqslant p_t(\omega) < p_2^*(1+z_{i(t-1)})$, then $\forall x_i \in \mathbb{Z}\bigcup\left[-z_{i(t-1)}(\omega),+\infty\right]$,

$$x_i \in argmax_{x\in\mathbb{Z}}\, V_{it\omega}(x|x_i),\ \ s.t.\ x \geqslant -z_{i(t-1)(\omega)} \Longleftrightarrow$$
$$\frac{q_1p_1^*(1+\eta)+q_2p_2^*(1+(1-q_3)\eta\lambda+q_3\eta)}{1+q_1\eta+q_2q_3\eta+q_3\eta\lambda+q_2(q_1+q_2)\eta\lambda} \leqslant p_t(\omega) \leqslant (q_1p_1^*+q_2p_2^*) + \frac{q_2q_3\eta(1-\lambda)(z_{i(t-1)}+x_i)p_2^*}{1+q_1\eta\lambda+(q_2+q_3)\eta}.$$

(6) If $p_t(\omega) \geqslant p_2^*(1+z_{i(t-1)})$, then $\forall x_i \in \mathbb{Z}\bigcup\left[-z_{i(t-1)}(\omega),+\infty\right]$,

$$x_i \in argmax_{x\in\mathbb{Z}}\, V_{it\omega}(x|x_i),\ \ s.t.\ x \geqslant -z_{i(t-1)(\omega)} \Longleftrightarrow$$
$$\frac{q_1p_1^*(1+\eta)+q_2p_2^*(1+\eta\lambda)+q_2q_3\eta(\lambda-1)(z_{i(t-1)}+x_i)p_2^*}{1+q_1\eta+(q_2+q_3)\eta\lambda} \leqslant p_t(\omega) \leqslant (q_1p_1^*+q_2p_2^*) + \frac{q_2q_3\eta(1-\lambda)(z_{i(t-1)}+x_i)p_2^*}{1+q_1\eta\lambda+(q_2+q_3)\eta}.$$

B Appendix: Proofs

PROOF OF CLAIM 1 We have shown that under reference-independent settings, the valuation for any risky asset of agent i in period t under state ω is her expected price in the next period $E_{it}(p_{t+1}(\omega)|s_{it}^{PX}(\omega))$. Then it suffices to prove that the clearing price $p_t(\omega) = p_t^*(\omega) \equiv max_i\ E_{it}(p_{t+1}(\omega)|s_{it}^{PX}(\omega))$. If $p_t(\omega) > p_t^*(\omega)$, then $\forall i$ she will choose to sell all the risky assets which violates market clearing; If $p_t(\omega) < p_t^*(\omega)$, for $j \in argmax_i E_{it}(p_{t+1}(\omega)|s_{it}^{PX}(\omega))$, $E_{jt}(p_{t+1}(\omega)|s_{jt}^{PX}(\omega)) > p_t(\omega)$ and she is willing to buy (nearly) infinite number of risky assets, which cannot constitute a REE as well. Thus, $p_t(\omega) = max_i\ E_{it}(p_{t+1}(\omega)|s_{it}^{PX}(\omega))$ and is thus well-defined.

PROOF OF PROPOSITION 1: We know that $\{\omega\} \subset s_{it}^{PX}(\omega) \subset s_{it'}^{PX}(\omega) = \{\omega\}$, $\forall t \geqslant t'$. Thus $s_{it}^{PX}(\omega) = \{\omega\}$. Suppose the plan (reference point) is $x_{it}(\omega)$, and we use $WTP_{it}^k(\omega)|x_{it}(\omega)$ ($WTA_{it}^k(\omega)|x_{it}(\omega)$) to denote i's willingness to pay (willingness to accept) for the kth unit of the risky asset in period t under ω with respect to $x_{it}(\omega)$. Firstly consider $WTP_{it}^k(\omega|x_{it}(\omega))$.

Buying $k-1$ units, $V_{it\omega}(k-1|x_{it}(\omega)) = m_{it} + (z_{i(t-1)} + k - 1)p_{t+1}(\omega) - (k-1)p_t(\omega) + \mu((x_{it}(\omega) - (k-1))(p_t(\omega) - p_{t+1}(\omega)))$. Denote $\mu((x_{it}(\omega) - (k-1))(p_t(\omega) - p_{t+1}(\omega)))$ as A_{k-1}.
Buying k units, $V_{it\omega}(k, WTP_{it}^k(\omega|x_{it}(\omega))|x_{it}(\omega)) = m_{it} + (z_{i(t-1)} + k)p_{t+1}(\omega) - (k-1)p_t(\omega) - WTP_{it}^k(\omega|x_{it}(\omega)) + \mu((x_{it}(\omega) - (k-1))(p_t(\omega) - p_{t+1}(\omega)) + p_{t+1}(\omega) - WTP_{it}^k(\omega|x_{it}(\omega)))$. Denote $\mu((x_{it}(\omega) - (k-1))(p_t(\omega) - p_{t+1}(\omega)) + p_{t+1}(\omega) - WTP_{it}^k(\omega|x_{it}(\omega)))$ as A_k.
By definition, the above two expressions must be equal to each other, but we must discuss the sign of A_k and A_{k-1} since there is a kink in $\mu(\cdot)$:

(1) If $A_{k-1} \leqslant 0$ and $A_k \leqslant 0$, then $WTP_{it}^k(\omega|x_{it}(\omega)) - p_{t+1}(\omega) = \eta\lambda(p_{t+1}(\omega) - WTP_{it}^k(\omega|x_{it}(\omega)))$ and thus $p_{t+1}(\omega) = WTP_{it}^k(\omega|x_{it}(\omega))$;

(2) If $A_{k-1} \geqslant 0$ and $A_k \geqslant 0$, then $WTP_{it}^k(\omega|x_{it}(\omega)) - p_{t+1}(\omega) = \eta(p_{t+1}(\omega) - WTP_{it}^k(\omega|x_{it}(\omega)))$ and thus $p_{t+1}(\omega) = WTP_{it}^k(\omega|x_{it}(\omega))$;

(3) If $A_{k-1} \leqslant 0$ and $A_k \geqslant 0$, then it must be the case that $WTP_{it}^k(\omega|x_{it}(\omega)) \leqslant p_{t+1}(\omega)$. Thus $0 \leqslant p_{t+1}(\omega) - WTP_{it}^k(\omega|x_{it}(\omega)) = \eta(\lambda-1)A_{k-1} - \eta(p_{t+1}(\omega) - WTP_{it}^k(\omega|x_{it}(\omega))) \leqslant -\eta(p_{t+1}(\omega) - WTP_{it}^k(\omega|x_{it}(\omega)))$ and $p_{t+1}(\omega) = WTP_{it}^k(\omega|x_{it}(\omega))$;

(4) If $A_{k-1} \geqslant 0$ and $A_k \leqslant 0$, then it must be the case that $WTP_{it}^k(\omega|x_{it}(\omega)) \geqslant p_{t+1}(\omega)$. Thus $0 \leqslant WTP_{it}^k(\omega|x_{it}(\omega)) - p_{t+1}(\omega) = \eta(\lambda-1)A_{k-1} + \eta\lambda(p_{t+1}(\omega) - WTP_{it}^k(\omega))$. This leads to $0 \leqslant WTP_{it}^k(\omega|x_{it}(\omega)) - p_{t+1}(\omega) = \frac{\eta(\lambda-1)}{1+\eta\lambda}A_{k-1}$. Note that $A_k \leqslant 0$, $A_{k-1} = A_k + WTP_{it}^k(\omega|x_{it}(\omega)) - p_{t+1}(\omega) \leqslant WTP_{it}^k(\omega|x_{it}(\omega)) - p_{t+1}(\omega)$ and $\frac{\eta(\lambda-1)}{1+\eta\lambda} < 1$. Then $WTP_{it}^k(\omega|x_{it}(\omega)) - p_{t+1}(\omega) \leqslant \frac{\eta(\lambda-1)}{1+\eta\lambda}(WTP_{it}^k(\omega|x_{it}(\omega)) - p_{t+1}(\omega))$ and $p_{t+1}(\omega) = WTP_{it}^k(\omega|x_{it}(\omega))$.

Thus we have $p_{t+1}(\omega) = WTP_{it}^{k}(\omega|x_{it}(\omega)) \;\; \forall t' \leqslant t \leqslant T, \forall k \in \mathbb{N}_{++}$. As for $WTA_{it}^{k}(\omega|x_{it}(\omega))$, the proof is virtually the same.

PROOF OF PROPOSITION 2 Firstly, notice that the condition $z_{i(t-1)} > 0$ is only to guarantee the feasibility of net trades $x_{it}(\omega)+1$ and $x_{it}(\omega)-1$. Then according to utility maximization, $V_{it\omega}(x_{it}(\omega)|x_{it}(\omega)) \geqslant V_{it\omega}(x_{it}(\omega)+1|x_{it}(\omega))$ and $V_{it\omega}(x_{it}(\omega)|x_{it}(\omega)) \geqslant V_{it\omega}(x_{it}(\omega)-1|x_{it}(\omega))$.

For the former inequality along with the definition of WTP, $\sum_{\omega' \in s_{it}^{PX}(\omega)} \sum_{\omega'' \in s_{it}^{PX}(\omega)} \pi_i(\omega')\pi_i(\omega'')u(m_{it}(\omega) + (z_{i(t-1)}(\omega)+x_{it}(\omega))p_{t+1}(\omega') - x_{it}(\omega)p_t(\omega)|m_{it}(\omega) + (z_{i(t-1)}(\omega)+x_{it}(\omega))p_{t+1}(\omega') - x_{it}(\omega)p_t(\omega)) =$
$\sum_{\omega' \in s_{it}^{PX}(\omega)} \sum_{\omega'' \in s_{it}^{PX}(\omega)} \pi_i(\omega')\pi_i(\omega'')u(m_{it}(\omega) + (z_{i(t-1)}(\omega)+x_{it}(\omega))p_{t+1}(\omega') - x_{it}(\omega)p_t(\omega) + p_{t+1}(\omega') - WTP_{it}^{1}(\omega|x_{it}(\omega))|m_{it}(\omega) + (z_{i(t-1)}(\omega)+x_{it}(\omega))p_{t+1}(\omega') - x_{it}(\omega)p_t(\omega)) \geqslant$
$\sum_{\omega' \in s_{it}^{PX}(\omega)} \sum_{\omega'' \in s_{it}^{PX}(\omega)} \pi_i(\omega')\pi_i(\omega'')u(m_{it}(\omega) + (z_{i(t-1)}(\omega)+x_{it}(\omega))p_{t+1}(\omega') - x_{it}(\omega)p_t(\omega) + p_{t+1}(\omega') - p_t(\omega)|m_{it}(\omega) + (z_{i(t-1)}(\omega)+x_{it}(\omega))p_{t+1}(\omega') - x_{it}(\omega)p_t(\omega))$. Observe that the difference of the last two expressions lies only in $-WTP_{it}^{1}(\omega|x_{it}(\omega))$ and $-p_t(\omega)$, both of which are constant in the computation. From Koszegi and Rabin (2007), $u(x|y) \leqslant u(x+c|y)$ where c is a constant if and only if $c \geqslant 0$, and thus we have $-WTP_{it}^{1}(\omega|x_{it}(\omega)) \geqslant -p_t(\omega)$. That is $WTP_{it}^{1}(\omega|x_{it}(\omega)) \leqslant p_t(\omega)$. Similarly, we can show that $WTA_{it}^{1}(\omega|x_{it}(\omega)) \geqslant p_t(\omega)$. Thus, $WTA_{it}^{1}(\omega|x_{it}(\omega)) \geqslant WTP_{it}^{*}(\omega)$.

PROOF OF PROPOSITION 3 For the necessity, the conclusion does not rely on the assumption of binary economies. For any given prospect and any given REE (P,X), if the price at ω in period t is not well-defined, then it must be the case that $0 \in argmax_{x \in \mathbb{Z}} \; V_{it\omega}(x|0), \;\; s.t. \; x \geqslant -z_{i(t-1)(\omega)}, \forall i$. Suppose by way of contradiction that $\exists \; i \neq j \;\; s.t. \; WTP_{it}^{1}(\omega) \geqslant WTA_{jt}^{1}(\omega)$. Denote the corresponding clearing price by $\hat{p}_t(\omega)$. Since silent market occurs in period t at state ω in any REE, it must be the case that $\hat{p}_t(\omega) \leqslant WTA_{st}^{1}(\omega)$ and $\hat{p}_t(\omega) \geqslant WTP_{st}^{1}(\omega)$ where $s = i \; or \; j$ otherwise agent i or j will strictly want to trade with the reference point of status quo and $x_{it}(\omega) = 0$ or $x_{jt}(\omega) = 0$ violates utility maximization. Then by **Proposition 2** and the assumption, we have $WTP_{it}^{1}(\omega) = WTA_{jt}^{1}(\omega) = \hat{p}_t(\omega)$. By definition, this implies $V_{it(\omega)}(1|0) = V_{it(\omega)}(0|0), V_{jt(\omega)}(-1|0) = V_{jt(\omega)}(0|0)$ and there is no strict loss in net trades and price is well-defined (or by DOA, the auctioneer will declare $p_t(\omega) = \hat{p}_t(\omega)$). Contradiction!

For sufficiency, we restrict to the binary prospect to make it more tractable. Suppose by way of contradiction that the price is observable in a specific REE as $p_t(\omega)$, then by definition, with status quo as the reference point, the silent market cannot strictly dominate all the non-zero profile of net trades. Since market clears, $\exists i, j \; s.t. \; x_i > 0, \; x_j < 0$ and $V_{it\omega}(x_i|0) \geqslant V_{it\omega}(0|0)$, $V_{jt\omega}(x_j|0) \geqslant V_{jt\omega}(0|0)$. Consider $V_{it\omega}(x_i|0) \geqslant V_{it\omega}(0|0)$. It can be shown that x_i will be cancelled out and we get $V_{it\omega}(1|0) \geqslant V_{it\omega}(0|0)$, which further implies $WTP_{it}^{1}(\omega) \geqslant p_t(\omega)$. Similarly, $V_{jt\omega}(x_j|0) \geqslant V_{jt\omega}(0|0)$ leads to $V_{jt\omega}(-1|0) \geqslant V_{jt\omega}(0|0)$ and $WTA_{jt}^{1}(\omega) \leqslant p_t(\omega)$. That is $\exists i \neq j \; s.t. \; WTA_{jt}^{1}(\omega) \leqslant WTP_{it}^{1}(\omega)$, again a contradiction with the condition $WTP_{it}^{1}(\omega) < WTA_{jt}^{1}(\omega), \; \forall i \neq j$.

PROOF OF PROPOSITION 4 From **Proposition 3**, if $x_{it}(\omega)=0$ does not constitute a *REE*, then price is observable and either $p_t(\omega)\leqslant WTP^1_{it}(\omega)=\frac{(1+\eta)q_{it\omega}p^{1*}_{i(t+1)\omega}+(1+\eta\lambda)(1-q_{it\omega})p^{2*}_{i(t+1)\omega}}{(1+\eta)q_{it\omega}+(1+\eta\lambda)(1-q_{it\omega})}$ or $p_t(\omega)\geqslant WTA^1_{it}(\omega)=q_{it\omega}p^{1*}_{i(t+1)\omega}+(1-q_{it\omega})p^{2*}_{i(t+1)\omega}$. Notice that only one of these two will happen.

(1) If $p_t(\omega)\leqslant WTP^1_{it}(\omega)<WTA^1_{it}(\omega)$, then agent has weak incentive to deviate from status quo by purchasing. Suppose by way of contradiction that $p_t(\omega)<WTP^1_{it}(\omega)$, and $x_{it}(\omega)$ appears in a *REE*, then $V_{it\omega}(x_{it}(\omega)+1|x_{it}(\omega))-V_{it\omega}(x_{it}(\omega)|x_{it}(\omega))$= $(1+\eta)q_{it\omega}p^{1*}_{i(t+1)\omega}+(1+\eta\lambda)(1-q_{it\omega})p^{2*}_{i(t+1)\omega}-p_t(\omega)((1+\eta)q_{it\omega}+(1+\eta\lambda)(1-q_{it\omega}))>0$. Notice that it is assumed that $m_i\gg max_{\omega\in\Omega}d(\omega)$, $x_{it}(\omega)+1$ will be affordable, contradicting with the fact that $x_{it}(\omega)$ maximizes i's utility in period t at ω. In this case, $p_t(\omega)=WTP^1_{it}(\omega)$. More specifically, for any feasible x_i, $V_{it\omega}(x_i|x_i)=V_{it\omega}(y_i|x_i)\ \ \forall y_i\geqslant x_i$ and $V_{it\omega}(x_i|x_i)>V_{it\omega}(y_i|x_i)\ \ \forall y_i<x_i$, that is, x_i satisfies utility-maximization for i. This offers intuition for the trading strategy.

(2) If $p_t(\omega)\geqslant WTA^1_{it}(\omega)>WTP^1_{it}(\omega)$, then agent has weak incentive to deviate from status quo by purchasing. If $p_t(\omega)>WTA^1_{it}(\omega)$, and $x_{it}(\omega)$ appears in a *REE*, then $V_{it\omega}(x_{it}(\omega)-1|x_{it}(\omega))-V_{it\omega}(x_{it}(\omega)|x_{it}(\omega))$= $(p_t(\omega)-(q_{it\omega}p^{1*}_{i(t+1)\omega}+(1-q_{it\omega})p^{2*}_{i(t+1)\omega}))((1+\eta)q_{it\omega}+(1+\eta\lambda)(1-q_{it\omega}))>0$. Then for any $x_{it}(\omega)>-z_{i(t-1)}(\omega)$, $x_{it}(\omega)-1$ will be feasible, contradicting with the fact that $x_{it}(\omega)$ maximizes i's utility in period t at ω. In this case, the only possible outcome is $x_{it}(\omega)=-z_{i(t-1)}(\omega)$, that is, the agent prefers short position. To show that this satisfies utility maximization, just verify $V_{it\omega}-z_{i(t-1)}(\omega)|-z_{i(t-1)}(\omega))>V_{it\omega}(y_i|-z_{i(t-1)}(\omega))\ \ \forall y_i\geqslant -z_{i(t-1)}(\omega)$.
If instead $p_t(\omega)=WTA^1_{it}(\omega)$, for any feasible x_i, $V_{it\omega}(x_i|x_i)=V_{it\omega}(y_i|x_i)\ \ \forall y_i\leqslant x_i$ and $V_{it\omega}(x_i|x_i)>V_{it\omega}(y_i|x_i)\ \ \forall y_i>x_i$, that is, x_i satisfies utility-maximization for i. Again, the agent weakly prefers short position.

PROOF OF PROPOSITION 5

The proof of existence is in fact an algorithm to find a *REE*:

(Step1) Assumed the initial information structure has been refined by prices and net trades ;

(Step2) For a given state ω and period t such that $max_iWTP^1_{it}(\omega)\leqslant min_iWTA^1_{it}(\omega)$, set $x_t(\omega)=0$. As for prices, if $\nexists i\neq j, s.t\ WTP^1_{it}(\omega)=WTA^1_{jt}(\omega)$, then set $p_t(\omega)=\emptyset$; if $\exists i\neq j, s.t\ WTP^1_{it}(\omega)=WTA^1_{jt}(\omega)$, then set $p_t(\omega)=max_iWTP^1_{it}(\omega)$;

(Step3) For a given state ω and period t such that $max_iWTP^1_{it}(\omega)>min_iWTA^1_{it}(\omega)$, set $p_t(\omega)=max_iWTP^1_{it}(\omega)$, then for those $i\notin argmax_iWTP^1_{it}(\omega)$, from **Proposition 4**, i is either indifferent (set $x_{it}(\omega)=0$) or chooses short position (set $x_{it}(\omega)=-z_{i(t-1)}(\omega)$). Then add up amounts of risky asset sold to S; for those $i\in argmax_iWTP^1_{it}(\omega)$, she is indifferent among purchasing any number of units and we assume that S is evenly divided with the extra units going to the agent with the smallest index;

(Step4) Check the tentative profiles of prices and net trades for condition (C1) and (C3). If they are satisfied, we are done; if not, denote the price-and-trade-refined information partitions as $\left\{S'_{it}\right\}_{t=1,2\cdots T}^{i=1,2\cdots I}$ and back to Step 1.

(Step5) The algorithm must terminate since no more information can be revealed if every information partition is a singleton. Notice that if so, $p_t(\omega) = d(\omega), \forall t, \forall \omega$ and $X = 0$ constitute a REE.

PROOF OF CLAIM 4 Suppose by way of contradiction that in a REE $p_t(\omega) > p^*$, that is, the spot price of the asset is strictly higher than the highest possible payoff in the next period. Intuitively, everyone will strictly prefers to sell at present and the market cannot clear.
Specifically, if $p_t(\omega) > p^*$, for any agent i with the equilibrium trade level x_i,
$V_{it\omega}(x_i - 1|x_i) = \frac{1}{\sum_{\omega' \in s_{it}^{PX}(\omega)} \pi(\omega')} \sum_{\omega' \in s_{it}^{PX}(\omega)} \sum_{\omega'' \in s_{it}^{PX}(\omega)} \pi_i(\omega')\pi_i(\omega'') u(m_{it} + (z_{i(t-1)} + x_i)p_{t+1}(\omega') - x_i p_t(\omega) + (p_t(\omega) - p_{t+1}(\omega'))|m_{it} + (z_{i(t-1)} + x_i)p_{t+1}(\omega'') - x_i p_t(\omega))$
$V_{it\omega}(x_i|x_i) = \frac{1}{\sum_{\omega' \in s_{it}^{PX}(\omega)} \pi(\omega')} \sum_{\omega' \in s_{it}^{PX}(\omega)} \sum_{\omega'' \in s_{it}^{PX}(\omega)} \pi_i(\omega')\pi_i(\omega'') u(m_{it} + (z_{i(t-1)} + x_i)p_{t+1}(\omega') - x_i p_t(\omega)|m_{it} + (z_{i(t-1)} + x_i)p_{t+1}(\omega'') - x_i p_t(\omega))$
Notice that $p_t(\omega) - p_{t+1}(\omega') \geqslant p_t(\omega) - p^* > 0$ and $u(x|y)$ is strictly increasing with x, for any ω' and ω'', we have $u(m_{it} + (z_{i(t-1)} + x_i)p_{t+1}(\omega') - x_i p_t(\omega) + (p_t(\omega) - p_{t+1}(\omega'))|m_{it} + (z_{i(t-1)} + x_i)p_{t+1}(\omega'') - x_i p_t(\omega)) > u(m_{it} + (z_{i(t-1)} + x_i)p_{t+1}(\omega') - x_i p_t(\omega)|m_{it} + (z_{i(t-1)} + x_i)p_{t+1}(\omega'') - x_i p_t(\omega))$, and thus $V_{it\omega}((x_i - 1|x_i) > V_{it\omega}((x_i|x_i)$. Due to utility-maximizing of REE trading profile, $x_i - 1$ must be unachievable for agent i, that is $x_i = -z_{i(t-1)}$. But this further implies that the market cannot clear as $\sum_i x_i < 0$.

PROOF OF PROPOSITION 3' For tractability, we will restrict our analysis to ternary prospect as discussed above. Specifically, consider the prospect $(p_1^*, q_1; p_2^*, q_2; 0, q_3 = 1 - q_1 - q_2)$ where $p_1^* > p_2^* > 0$. Then, by the same argument as the binary case, we can safely assume that $0 \leqslant p_t(\omega) \leqslant p_1^*$.
For the first part of the proposition, if suffices to show that $V_{it\omega}(x_i|0) \geqslant V_{it\omega}(0|0) \longrightarrow V_{it\omega}(1|0) \geqslant V_{it\omega}(0|0)$ for $x_i \in \mathbb{N}_+$, and $V_{it\omega}(-x_i|0) \geqslant V_{it\omega}(0|0) \longrightarrow V_{it\omega}(-1|0) \geqslant V_{it\omega}(0|0)$ for $x_i \in \mathbb{N}_+$. We first consider the case of purchasing.

$$
\begin{aligned}
V_{it\omega}(x_i|0) = m_{it} &+ (z_{i(t-1)} + x_i)(q_1 p_1^* + q_2 p_2^*) - x_i p_t(\omega) \\
&+ q_1^2 \mu(x_i(p_1^* - p_t(\omega))) \qquad \longrightarrow \text{positive since } x_i(p_1^* - p_t(\omega)) \geqslant 0 \\
&+ q_1 q_2 \mu((z_{i(t-1)} + x_i)p_1^* - x_i p_t(\omega) - z_{i(t-1)} p_2^*) \longrightarrow \text{positive since } p_1^* > p_2^*, p_1^* \geqslant p_t(\omega) \\
&+ q_1 q_2 \mu((z_{i(t-1)} + x_i)p_2^* - x_i p_t(\omega) - z_{i(t-1)} p_1^*) \qquad \longrightarrow \text{ambiguous sign}
\end{aligned}
$$

$$
\begin{aligned}
&+q_2^2\mu(x_i(p_2^*-p_t(\omega))) && \longrightarrow \text{ ambiguous sign}\\
&+q_2q_3\mu((z_{i(t-1)}+x_i)p_2^*-x_ip_t(\omega)) && \longrightarrow \text{ ambiguous sign}\\
&+q_2q_3\mu(-z_{i(t-1)}p_2^*-x_ip_t(\omega)) && \longrightarrow \text{ negative since } x_ip_t(\omega)\geqslant 0, z_{i(t-1)}p_2^*\geqslant 0\\
&+q_3^2\mu(-x_ip_t(\omega)) && \longrightarrow \text{ negative since } x_ip_t(\omega)\geqslant 0\\
&+q_1q_3\mu((z_{i(t-1)}+x_i)p_1^*-x_ip_t(\omega)) && \longrightarrow \text{ positive since } x_i(p_1^*-p_t(\omega))\geqslant 0\\
&+q_2q_3\mu(-z_{i(t-1)}p_1^*-x_ip_t(\omega)) && \longrightarrow \text{ negative since } x_ip_t(\omega)\geqslant 0, z_{i(t-1)}p_1^*\geqslant 0
\end{aligned}
$$

$$
\begin{aligned}
V_{it\omega}(0|0) = {} & m_{it}+z_{i(t-1)}(q_1p_1^*+q_2p_2^*)\\
&+q_1q_2\mu(z_{i(t-1)}(p_1^*-p_2^*))+q_1q_2\mu(-z_{i(t-1)}(p_1^*-p_2^*))\\
&+q_2q_3\mu(z_{i(t-1)}p_2^*)+q_2q_3\mu(-z_{i(t-1)}p_2^*)\\
&+q_1q_3\mu(z_{i(t-1)}p_1^*)+q_2q_3\mu(-z_{i(t-1)}p_1^*)\\
= {} & \eta(1-\lambda)z_{i(t-1)}(q_1q_2(p_1^*-p_2^*)+q_1q_3p_1^*+q_2q_3p_2^*)
\end{aligned}
$$

Denote $A1\equiv(z_{i(t-1)}+x_i)p_2^*-x_ip_t(\omega)-z_{i(t-1)}p_1^*$, $A2\equiv x_i(p_2^*-p_t(\omega))$, $A3\equiv(z_{i(t-1)}+x_i)p_2^*-x_ip_t(\omega)$ and clearly, $A1\leqslant A2\leqslant A3$.

Then $V_{it\omega}(x_i|0)-V_{it\omega}(0|0)$ reduces to $x_i(q_1p_1^*+q_2p_2^*-p_t(\omega))+q_1^2\eta x_i(p_1^*-p_t(\omega))+q_1q_2\eta x_i(p_1^*-p_t(\omega))+q_1q_2(\mu(A1)+\eta\lambda z_{i(t-1)}(p_1^*-p_2^*))+q_2^2\mu(A2)+q_2q_3(\mu(A3)-\eta z_{i(t-1)}p_2^*)-q_2q_3\eta\lambda x_ip_t(\omega)-q_3^2\eta\lambda x_ip_t(\omega)+q_1q_3\eta x_i(p_1^*-p_t(\omega))-q_1q_3\eta\lambda x_ip_t(\omega)$

Now it is time to discuss over the range of spot prices.

(1). If $p_t(\omega)>p_2^*(1+\frac{z_{i(t-1)}}{x_i})$, that is, $A1\leqslant A2\leqslant A3<0$,

then $V_{it\omega}(x_i|0)\geqslant V_{it\omega}(0|0)\Longleftrightarrow$

$x_i(q_1p_1^*+q_2p_2^*-p_t(\omega))+q_1^2\eta x_i(p_1^*-p_t(\omega))+q_1q_2\eta x_i(p_1^*-p_t(\omega))+q_1q_2\eta\lambda(p_2^*-p_t(\omega))+q_2^2\eta\lambda x_i(p_2^*-p_t(\omega))+q_2q_3\eta\lambda x_i((p_2^*-p_t(\omega))-q_2q_3\eta\lambda x_ip_t(\omega)-q_3^2\eta\lambda x_ip_t(\omega)+q_1q_3\eta x_i(p_1^*-p_t(\omega))-q_1q_3\eta\lambda x_ip_t(\omega)\geqslant q_2q_3\eta(1-\lambda)z_{i(t-1)}p_2^*\Longleftrightarrow$

$x_i(q_1p_1^*(1+\eta)+q_2p_2^*(1+\eta\lambda)-p_t(\omega)(1+q_1\eta+(q_2+q_3)\eta\lambda))\geqslant q_2q_3\eta(1-\lambda)z_{i(t-1)}p_2^*\quad(*)$

Now think about $V(1|0)-V(0|0)$, that is $x_i=1$, and correspondingly, $A2'\leqslant 0$ and $A1'\leqslant 0$ since $p_t(\omega)>p_2^*(1+\frac{z_{i(t-1)}}{x_i})\geqslant p_2^*$. However, the sign of $A3'$ is uncertain.

(i) If still, $p_t(\omega)\geqslant p_2^*(1+z_{i(t-1)})$, then $A3'\leqslant 0$ and,
$V(1|0)-V(0|0)=q_1p_1^*(1+\eta)+q_2p_2^*(1+\eta\lambda)-p_t(\omega)(1+q_1\eta+(q_2+q_3)\eta\lambda)+q_2q_3\eta(\lambda-1)z_{i(t-1)}p_2^*\geqslant(1-\frac{1}{x_i})q_2q_3\eta(\lambda-1)z_{i(t-1)}p_2^*\geqslant 0$ due to $(*)$ and $x_i\in\mathbb{N}_+$.

(ii) If $p_2^*(1+\frac{z_{i(t-1)}}{x_i})<p_t(\omega)<p_2^*(1+z_{i(t-1)})$, then $A3'\geqslant 0$.
Since $A3\leqslant 0$, $\mu(A3)=\eta\lambda((z_{i(t-1)}+x_i)p_2^*-x_ip_t(\omega))\leqslant\eta((z_{i(t-1)}+x_i)p_2^*-x_ip_t(\omega))\leqslant 0$.
Then $0\leqslant V(x_i|0)-V(0|0)\leqslant x_i(q_1p_1^*(1+\eta)+q_2p_2^*(1+(1-q_3)\eta\lambda+q_3\eta)-p_t(\omega)(1+q_1\eta+q_2q_3\eta+q_3\eta\lambda+q_2(q_2+q_1)\eta\lambda))$
and thus $V(1|0)-V(0|0)=q_1p_1^*(1+\eta)+q_2p_2^*(1+(1-q_3)\eta\lambda+q_3\eta)-p_t(\omega)(1+q_1\eta+q_2q_3\eta+q_3\eta\lambda+q_2(q_2+q_1)\eta\lambda)\geqslant 0$ since $x_i>0$.

Accordingly, $V(x_i|0)\geqslant V(0|0)\longrightarrow V(1|0)\geqslant V(0|0)$ when $p_t(\omega)>p_2^*(1+\frac{z_{i(t-1)}}{x_i})$.

(2). If $p_t(\omega) < p_2^* + \frac{z_{i(t-1)}}{x_i}(p_2^* - p_1^*) \leqslant p_2^*$, that is, $0 < A1 \leqslant A2 \leqslant A3$,
then $V_{it\omega}(x_i|0) \geqslant V_{it\omega}(0|0) \Longleftrightarrow$
$x_i(q_1p_1^* + q_2p_2^* - p_t(\omega)) + q_1^2\eta x_i(p_1^* - p_t(\omega)) + q_1q_2\eta x_i(p_1^* - p_t(\omega)) + q_1q_2\eta(p_2^* - p_t(\omega)) + q_2^2\eta x_i(p_2^* - p_t(\omega)) + q_2q_3\eta x_i((p_2^* - p_t(\omega)) - q_2q_3\eta\lambda x_i p_t(\omega) - q_3^2\eta\lambda x_i p_t(\omega) + q_1q_3\eta x_i(p_1^* - p_t(\omega)) - q_1q_3\eta\lambda x_i p_t(\omega) \geqslant q_1q_2\eta(1-\lambda)z_{i(t-1)}(p_1^* - p_2^*) \Longleftrightarrow$
$x_i((q_1p_1^* + q_2p_2^*)(1+\eta) - p_t(\omega)(1 + (q_1+q_2)\eta + q_3\eta\lambda)) \geqslant q_1q_2\eta(1-\lambda)z_{i(t-1)}(p_1^* - p_2^*) \quad (**)$
Now think about $V(1|0) - V(0|0)$, and correspondingly, $A2' \geqslant 0$ and $A3' \geqslant 0$ since $p_t(\omega) > p_2^*(1 + \frac{z_{i(t-1)}}{x_i}) \geqslant p_2^*$. However, the sign of $A1'$ is uncertain.

(i) If still, $p_t(\omega) \leqslant p_2^* + z_{i(t-1)}(p_2^* - p_1^*)$, then $A1' \geqslant 0$ and,
$V(1|0) - V(0|0) = (q_1p_1^* + q_2p_2^*)(1+\eta) - p_t(\omega)(1 + (q_1+q_2)\eta + q_3\eta\lambda) + q_1q_2\eta(\lambda-1)z_{i(t-1)}(p_1^* - p_2^*) \geqslant (1 - \frac{1}{x_i})q_1q_2\eta(\lambda-1)z_{i(t-1)}(p_1^* - p_2^*) \geqslant 0$ due to $(**)$ and $x_i \in \mathbb{N}_+$.

(ii) If $p_2^* + z_{i(t-1)}(p_2^* - p_1^*) < p_t(\omega) < p_2^* + \frac{z_{i(t-1)}}{x_i}(p_2^* - p_1^*)$, then $A1' \leqslant 0$.
Since $A1 \geqslant 0$, $0 \leqslant \mu(A1) = \eta((z_{i(t-1)} + x_i)p_2^* - x_i p_t(\omega) - z_{i(t-1)}p_1^*) \leqslant \eta\lambda((z_{i(t-1)} + x_i)p_2^* - x_i p_t(\omega) - z_{i(t-1)}p_1^*)$.
Then $0 \leqslant V(x_i|0) - V(0|0) \leqslant x_i(q_1p_1^*(1+\eta) + q_2p_2^*(1 + (1-q_1)\eta + q_1\eta\lambda) - p_t(\omega)(1 + q_1\eta + q_2(q_2+q_3)\eta + q_3\eta\lambda + q_2q_1\eta\lambda))$
and thus $V(1|0) - V(0|0) = q_1p_1^*(1+\eta) + q_2p_2^*(1 + (1-q_1)\eta + q_1\eta\lambda) - p_t(\omega)(1 + q_1\eta + q_2(q_2+q_3)\eta + q_3\eta\lambda + q_2q_1\eta\lambda) \geqslant 0$ since $x_i > 0$.

Accordingly, $V(x_i|0) \geqslant V(0|0) \longrightarrow V(1|0) \geqslant V(0|0)$ when $p_t(\omega) < p_2^* + \frac{z_{i(t-1)}}{x_i}(p_2^* - p_1^*)$.

(3).If $p_2^* + \frac{z_{i(t-1)}}{x_i}(p_2^* - p_1^*) \leqslant p_t(\omega) \leqslant p_2^*$, then $A3 \geqslant 0, A1 \leqslant 0$ and $A2 \geqslant 0$. Directly, we have $p_2^* + z_{i(t-1)}(p_2^* - p_1^*) \leqslant p_t(\omega) \leqslant p_2^*$ and thus $A3' \geqslant 0, A1' \leqslant 0$ and $A2' \geqslant 0$.
$V_{it\omega}(x_i|0) \geqslant V_{it\omega}(0|0) \Longleftrightarrow$
$x_i(q_1p_1^*(1+\eta) + q_2p_2^*(1 + (1-q_1)\eta + q_1\eta\lambda) - p_t(\omega)(1 + q_1\eta + q_2(q_2+q_3)\eta + q_3\eta\lambda + q_2q_1\eta\lambda)) \geqslant 0 \Longleftrightarrow V_{it\omega}(1|0) \geqslant V_{it\omega}(0|0)$

(4) If $p_2^* \leqslant p_t(\omega) \leqslant p_2^*(1 + \frac{z_{i(t-1)}}{x_i})$, then then $A3 \geqslant 0, A1 \leqslant 0$ and $A2 \leqslant 0$. Directly, we have $p_2^* \leqslant p_t(\omega) \leqslant p_2^*(1 + z_{i(t-1)})$ and thus $A3' \geqslant 0, A1' \leqslant 0$ and $A2' \leqslant 0$.
$V_{it\omega}(x_i|0) \geqslant V_{it\omega}(0|0) \Longleftrightarrow$
$x_i(q_1p_1^*(1+\eta) + q_2p_2^*(1 + (1-q_3)\eta\lambda + q_3\eta) - p_t(\omega)(1 + q_1\eta + q_2q_3\eta + q_3\eta\lambda + q_2(q_2+q_1)\eta\lambda)) \geqslant 0 \Longleftrightarrow V_{it\omega}(1|0) \geqslant V_{it\omega}(0|0)$

Now we are done with the case where $x_i > 0$ and the situation of selling should be similar. But to acquire the expression of WTP, we will as well go into details. Similarly we have $0 \leqslant p_t(\omega) \leqslant p_1^*$

and short sales constraint implies $x_i < z_{i(t-1)}$.

$$
\begin{aligned}
V_{it\omega}(-x_i|0) = m_{it} &+ (z_{i(t-1)} - x_i)(q_1 p_1^* + q_2 p_2^*) + x_i p_t(\omega) \\
&+ q_1^2 \mu(-x_i(p_1^* - p_t(\omega))) \quad \longrightarrow \text{ negative since } x_i(p_1^* - p_t(\omega)) \geqslant 0 \\
&+ q_1 q_2 \mu((z_{i(t-1)} - x_i)p_1^* + x_i p_t(\omega) - z_{i(t-1)} p_2^*) \quad \longrightarrow \text{ ambiguous sign} \\
&+ q_1 q_2 \mu((z_{i(t-1)} - x_i)p_2^* + x_i p_t(\omega) - z_{i(t-1)} p_1^*) \quad \longrightarrow \text{negative since } p_1^* > p_2^*, p_1^* \geqslant p_t(\omega) \\
&+ q_2^2 \mu(-x_i(p_2^* - p_t(\omega))) \quad \longrightarrow \text{ ambiguous sign} \\
&+ q_2 q_3 \mu((z_{i(t-1)} - x_i)p_2^* + x_i p_t(\omega)) \quad \longrightarrow \text{ positive since } x_i p_t(\omega) \geqslant 0, z_{i(t-1)} \geqslant x_i \\
&+ q_2 q_3 \mu(-z_{i(t-1)} p_2^* + x_i p_t(\omega)) \quad \longrightarrow \text{ ambiguous sign} \\
&+ q_3^2 \mu(x_i p_t(\omega)) \quad \longrightarrow \text{ positive since } x_i p_t(\omega) \geqslant 0 \\
&+ q_1 q_3 \mu((z_{i(t-1)} - x_i)p_1^* + x_i p_t(\omega)) \quad \longrightarrow \text{ positive since } z_{i(t-1)} \geqslant x_i \\
&+ q_2 q_3 \mu(-z_{i(t-1)} p_1^* + x_i p_t(\omega)) \quad \longrightarrow \text{ negative since } z_{i(t-1)} \geqslant x_i \text{ and } p_1^* \geqslant p_t(\omega)
\end{aligned}
$$

Denote $B1 \equiv (z_{i(t-1)} - x_i)p_1^* + x_i p_t(\omega) - z_{i(t-1)} p_2^*$, $B2 \equiv -x_i(p_2^* - p_t(\omega))$, $B3 \equiv x_i p_t(\omega) - z_{i(t-1)} p_2^*$.

$$B1 \geqslant 0 \Longleftrightarrow p_t(\omega) \geqslant p_1^* + \frac{z_{i(t-1)}}{x_i}(p_2^* - p_1^*)$$

$$B2 \geqslant 0 \Longleftrightarrow p_t(\omega) \geqslant p_2^*$$

$$B3 \geqslant 0 \Longleftrightarrow p_t(\omega) \geqslant \frac{z_{i(t-1)}}{x_i} p_2^*$$

And since $0 < x_i \leqslant z_{i(t-1)}$, $\frac{z_{i(t-1)}}{x_i} \geqslant 1$ and $\frac{z_{i(t-1)}}{x_i} p_2^* \geqslant p_2^* \geqslant p_1^* + \frac{z_{i(t-1)}}{x_i}(p_2^* - p_1^*)$.

(1). If $p_t(\omega) < p_1^* + \frac{z_{i(t-1)}}{x_i}(p_2^* - p_1^*)$, that is, $B1 \leqslant 0, B2 \leqslant 0, B3 \leqslant 0$,

then $V_{it\omega}(-x_i|0) \geqslant V_{it\omega}(0|0) \Longleftrightarrow$

$x_i(p_t(\omega) - q_1 p_1^* + q_2 p_2^*)(1 + (q_1 + q_2)\eta\lambda + q_3\eta) \geqslant q_1 q_2 \eta(1-\lambda) z_{i(t-1)}(p_1^* - p_2^*) \quad (*)$

Now think about $V(-1|0) - V(0|0)$, that is $x_i = 1$, and correspondingly, $B2' \leqslant 0$ and $B3' \leqslant 0$ since $p_t(\omega) \leqslant p_2^*$. However, the sign of $B1'$ is uncertain.

(i) If still, $p_t(\omega) \leqslant p_1^* + z_{i(t-1)}(p_2^* - p_1^*)$, then $B1' \leqslant 0$ and,
$V(1|0) - V(0|0) = p_t(\omega) - q_1 p_1^* + q_2 p_2^*)(1 + (q_1 + q_2)\eta\lambda + q_3\eta - q_1 q_2 \eta(1-\lambda) z_{i(t-1)}(p_1^* - p_2^*) \geqslant (1 - \frac{1}{x_i}) q_1 q_2 \eta(\lambda - 1) z_{i(t-1)}(p_1^* - p_2^*) \geqslant 0$ due to $(*)$ and $x_i \in \mathbb{N}_+$.

(ii) If $p_1^* + z_{i(t-1)}(p_2^* - p_1^*) < p_t(\omega) < p_1^* + \frac{z_{i(t-1)}}{x_i}(p_2^* - p_1^*)$, then $B1' \geqslant 0$.
Since $B1 \leqslant 0$, $\mu(B1) = \eta\lambda((z_{i(t-1)} - x_i)p_1^* + x_i p_t(\omega) - z_{i(t-1)} p_2^*) \leqslant \eta((z_{i(t-1)} - x_i)p_1^* + x_i p_t(\omega) - z_{i(t-1)} p_2^*) \leqslant 0$.
Then $0 \leqslant V(-x_i|0) - V(0|0) \leqslant x_i(-q_1 p_1^*(1 + q_1\eta\lambda + (q_2 + q_3)\eta) - q_2 p_2^*(1 + (q_1 + q_2)\eta\lambda + q_3\eta) + p_t(\omega)(1 + q_1\eta\lambda + q_3\eta + q_2(q_2 + q_3)\eta\lambda + q_1 q_2 \eta))$
and thus $V(-1|0) - V(0|0) = -q_1 p_1^*(1 + q_1\eta\lambda + (q_2 + q_3)\eta) - q_2 p_2^*(1 + (q_1 + q_2)\eta\lambda + q_3\eta) + p_t(\omega)(1 + q_1\eta\lambda + q_3\eta + q_2(q_2 + q_3)\eta\lambda + q_1 q_2 \eta) \geqslant 0$ since $x_i > 0$.

Accordingly, $V(-x_i|0) \geqslant V(0|0) \longrightarrow V(-1|0) \geqslant V(0|0)$ when $p_t(\omega) < p_1^* + \frac{z_{i(t-1)}}{x_i}(p_2^* - p_1^*)$.

(2). If $p_t(\omega) > p_2^*\frac{z_{i(t-1)}}{x_i} \geqslant p_2^*$, that is, $B1 \geqslant 0, B2 \geqslant 0, B3 \geqslant 0$,
then $V_{it\omega}(-x_i|0) \geqslant V_{it\omega}(0|0) \Longleftrightarrow$
$x_i((p_t(\omega) - q_1p_1^* - q_2p_2^*)(1 + q_1\eta\lambda + (q_2 + q_3)\eta) \geqslant q_2q_3\eta(1-\lambda)z_{i(t-1)}p_2^* \quad (**)$
Now think about $V(-1|0) - V(0|0)$, and correspondingly, $B2' \geqslant 0$ and $B1' \geqslant 0$ since $p_t(\omega) > p_2^*\frac{z_{i(t-1)}}{x_i} \geqslant p_2^*$. However, the sign of $B3'$ is uncertain.

(i) If still, $p_t(\omega) \geqslant p_2^* z_{i(t-1)}$, then $B3' \geqslant 0$ and,
$V(-1|0) - V(0|0) = ((p_t(\omega) - q_1p_1^* - q_2p_2^*)(1 + q_1\eta\lambda + (q_2+q_3)\eta) + q_2q_3\eta(\lambda-1)z_{i(t-1)}p_2^* \geqslant (1-\frac{1}{x_i})q_2q_3\eta(\lambda-1)z_{i(t-1)}p_2^* \geqslant 0$ due to $(**)$ and $x_i \in \mathbb{N}_+$.

(ii) If $p_2^*\frac{z_{i(t-1)}}{x_i} < p_t(\omega) < p_2^* z_{i(t-1)}$, then $B3' \leqslant 0$.
Since $B3 \geqslant 0$, $0 \leqslant \mu(B3) = \eta(x_ip_t(\omega) - z_{i(t-1)}p_2^*) \leqslant \eta\lambda(x_ip_t(\omega) - z_{i(t-1)}p_2^*)$.
Then $0 \leqslant V(-x_i|0) - V(0|0) \leqslant x_i((-q_1p_1^* - q_2p_2^*)(1 + q_1\eta\lambda + (q_2+q_3)\eta) - p_t(\omega)(1 + q_1\eta\lambda + (q_2+q_3)\eta + q_2q_3\eta(\lambda-1)))$
and thus $V(-1|0) - V(0|0) = (-q_1p_1^* - q_2p_2^*)(1 + q_1\eta\lambda + (q_2+q_3)\eta) - p_t(\omega)(1 + q_1\eta\lambda + (q_2+q_3)\eta + q_2q_3\eta(\lambda-1)) \geqslant 0$ since $x_i > 0$.

Accordingly, $V(-x_i|0) \geqslant V(0|0) \longrightarrow V(-1|0) \geqslant V(0|0)$ when $p_t(\omega) > p_2^*\frac{z_{i(t-1)}}{x_i}$.

(3).If $p_1^* + \frac{z_{i(t-1)}}{x_i}(p_2^* - p_1^*) \leqslant p_t(\omega) \leqslant p_2^*$, then $B3 \leqslant 0, B1 \geqslant 0$ and $B2 \leqslant 0$. Directly, we have $p_1^* + z_{i(t-1)}(p_2^* - p_1^*) \leqslant p_t(\omega) \leqslant p_2^*$ and thus $B3' \leqslant 0, B1' \geqslant 0$ and $B2' \leqslant 0$.
$V_{it\omega}(-x_i|0) \geqslant V_{it\omega}(0|0) \Longleftrightarrow$
$x_i(-q_1p_1^*(1 + q_1\eta\lambda + (q_2+q_3)\eta) - q_2p_2^*(1 + q_3\eta + (q_1+q_2)\eta\lambda) - p_t(\omega)(1 + q_1\eta\lambda + q_2(q_2 + q_3)\eta\lambda + q_3\eta + q_2q_1\eta)) \geqslant 0 \Longleftrightarrow V_{it\omega}(-1|0) \geqslant V_{it\omega}(0|0)$

(4) If $p_2^* \leqslant p_t(\omega) \leqslant p_2^*\frac{z_{i(t-1)}}{x_i}$, then then $B3 \leqslant 0, B1 \geqslant 0$ and $B2 \geqslant 0$. Directly, we have $p_2^* \leqslant p_2^*\frac{z_{i(t-1)}}{x_i}$ and thus $B3' \leqslant 0, B1' \geqslant 0$ and $B2' \geqslant 0$.
$V_{it\omega}(-x_i|0) \geqslant V_{it\omega}(0|0) \Longleftrightarrow$
$x_i((-q_1p_1^* - q_2p_2^*)(1 + q_1\eta\lambda + (q_2+q_3)\eta) - p_t(\omega)(1 + q_1\eta\lambda + (q_2+q_3)\eta + q_2q_3\eta(\lambda-1))) \geqslant 0 \Longleftrightarrow V_{it\omega}(-1|0) \geqslant V_{it\omega}(0|0)$
Now we have proved the first part in **Proposition 3'**. As for the second part, necessity is given in **Proposition 3** without the assumption of binary economy. Assume $WTP_{it}^1(\omega) < WTA_{jt}^1(\omega)$, $\forall i \neq j$. Suppose by way of contradiction that the price is observable in a specific REE as $p_t(\omega)$, then by definition, with status quo as the reference point, the silent market cannot strictly dominate all the non-zero profile of net trades. Since market clears, $\exists i, j$ $s.t.$ $x_i > 0$, $x_j < 0$ and $V_{it\omega}(x_i|0) \geqslant V_{it\omega}(0|0)$, $V_{jt\omega}(x_j|0) \geqslant V_{jt\omega}(0|0)$. Consider $V_{it\omega}(x_i|0) \geqslant V_{it\omega}(0|0)$. From the first part, $V_{it\omega}(1|0) \geqslant V_{it\omega}(0|0)$, which further implies $WTP_{it}^1(\omega) \geqslant p_t(\omega)$. Similarly, $V_{jt\omega}(x_j|0) \geqslant V_{jt\omega}(0|0)$ leads to

$V_{jt\omega}(-1|0) \geqslant V_{jt\omega}(0|0)$ and $WTA^1_{jt}(\omega) \leqslant p_t(\omega)$. That is $\exists i \neq j\ s.t.\ WTA^1_{jt}(\omega) \leqslant WTP^1_{it}(\omega)$, again a contradiction with the condition $WTP^1_{it}(\omega) < WTA^1_{jt}(\omega)$, $\forall i \neq j$.

PROOF OF COROLLARY 3 Similarly, we use ternary prospects for illustration. Notice that $WTP^1_{it}(\omega|x_i) \leqslant p_t(\omega) \leqslant WTA^1_{it}(\omega|x_i) \xLeftrightarrow{by\ definition} V_{it\omega}(x_i|x_i) \geqslant V_{it\omega}(x_i+1|x_i)$ and $V_{it\omega}(x_i|x_i) \geqslant V_{it\omega}(x_i-1|x_i)$. Thus, necessity comes directly.

For sufficiency, we show that the case with arbitrary reference point can be reduced to one with reference point as status quo. By simple calculation, for $k \in \mathbb{N}_+$,

$$
\begin{aligned}
V_{it\omega}(x_i+k|x_i) - V_{it\omega}(x_i|x_i) = {} & k(q_1p_1^* + q_2p_2^* - p_t(\omega)) \\
& + q_1^2\mu(k(p_1^* - p_t(\omega))) \\
& + q_1q_2\mu((z_{i(t-1)} + x_i + k)p_1^* - kp_t(\omega) - (z_{i(t-1)} + x_i)p_2^*) \\
& + q_1q_2\mu((z_{i(t-1)} + x_i + k)p_2^* - kp_t(\omega) - (z_{i(t-1)} + x_i)p_1^*) \\
& + q_2^2\mu(k(p_2^* - p_t(\omega))) \\
& + q_2q_3\mu((z_{i(t-1)} + x_i + k)p_2^* - kp_t(\omega)) \\
& + q_2q_3\mu((-z_{i(t-1)} + x_i)p_2^* - kp_t(\omega)) \\
& + q_3^2\mu(-kp_t(\omega)) \\
& + q_1q_3\mu((z_{i(t-1)} + x_i + k)p_1^* - kp_t(\omega)) \\
& + q_2q_3\mu((-z_{i(t-1)} + x_i)p_1^* - kp_t(\omega)) \\
& + q_1q_2\mu((z_{i(t-1)} + x_i)(p_1^* - p_2^*)) + q_1q_2\mu(-(z_{i(t-1)} + x_i)(p_1^* - p_2^*)) \\
& + q_1q_3\mu((z_{i(t-1)} + x_i)p_1^*) + q_1q_3\mu(-(z_{i(t-1)} + x_i)p_1^*) \\
& + q_2q_3\mu((z_{i(t-1)} + x_i)p_2^*) + q_2q_3\mu(-(z_{i(t-1)} + x_i)p_2^*)
\end{aligned}
$$

By simplicity, we relabel $z_{i(t-1)} + x_i$ and k as "$z_{i(t-1)}$" and "x_i", those used in the proof of **Proposition 3'** and then $V_{it\omega}(x_i+k|x_i) - V_{it\omega}(x_i|x_i)$ reduces to $V_{it\omega}(k|0) - V_{it\omega}(0|0)$. Intuitively, in a *REE*, the plan is expected to be carried out and the agent will care about the anticipated position instead of the status quo. That is, in some sense we can treat the plan as having already been carried out.

By replacing the weak inequality in the conditions and proof of **Proposition 3'** with strict inequality, $V_{it\omega}(k|0) > V_{it\omega}(0|0) \Longrightarrow V_{it\omega}(1|0) > V_{it\omega}(0|0)$. Then the contraposition implies, $V_{it\omega}(0|0) \geqslant V_{it\omega}(1|0) \Longrightarrow V_{it\omega}(0|0) \geqslant V_{it\omega}(k|0)$. Similarly, $V_{it\omega}(0|0) \geqslant V_{it\omega}(-1|0) \Longrightarrow V_{it\omega}(0|0) \geqslant V_{it\omega}(-k|0)$. Thus, for all feasible $k \in \mathbb{N}_+$,

$V_{it\omega}(x_i|x_i) \geqslant V_{it\omega}(x_i+1|x_i) \Longrightarrow V_{it\omega}(x_i|x_i) \geqslant V_{it\omega}(x_i+k|x_i)$

$V_{it\omega}(x_i|x_i) \geqslant V_{it\omega}(x_i-1|x_i) \Longrightarrow V_{it\omega}(x_i|x_i) \geqslant V_{it\omega}(x_i-k|x_i)$

That is, $WTP^1_{it}(\omega|x_i) \leqslant p_t(\omega) \leqslant WTA^1_{it}(\omega|x_i) \Longrightarrow V_{it\omega}(x|x_i) \leqslant V_{it\omega}(x_i|x_i)$ and sufficiency follows.

PROOF OF CLAIM 5 Consider the ternary prospect $(p_1^*, q_1; p_2^*, q_2; 0, q_3 = 1 - q_1 - q_2)$.

(1). If $p_t(\omega) \geqslant p_2^*(1 + z_{i(t-1)})$,

then $V_{it\omega}(1|0) > V_{it\omega}(0|0) \Longleftrightarrow p_t(\omega) < \frac{q_1p_1^*(1+\eta) + q_2p_2^*(1+\eta\lambda) + q_2q_3\eta(\lambda-1)z_{i(t-1)}p_2^*}{1 + q_1\eta + (q_2+q_3)\eta\lambda}$. $(***)$

Consider $V_{it\omega}(x_i+1|x_i), V_{it\omega}(x_i|x_i)$.

(i) If still, $p_t(\omega) \geqslant p_2^*(1+x_i+z_{i(t-1)})$, then
$V_{it\omega}(x_i+1|x_i)) > V_{it\omega}(x_i|x_i) \Longleftrightarrow p_t(\omega) < \frac{q_1p_1^*(1+\eta)+q_2p_2^*(1+\eta\lambda)+q_2q_3\eta(\lambda-1)(z_{i(t-1)}+x_i)p_2^*}{1+q_1\eta+(q_2+q_3)\eta\lambda}$.
The latter inequality directly comes from $(***)$.

(ii) If $p_2^*(1+z_{i(t-1)}) \leqslant p_t(\omega) < p_2^*(1+x_i+z_{i(t-1)})$, then $\mu((z_{i(t-1)}+1)p_2^*-p_t(\omega)) = \eta\lambda((z_{i(t-1)}+1)p_2^*-p_t(\omega)) \leqslant \eta((z_{i(t-1)}+1)p_2^*-p_t(\omega)) \leqslant 0$.
Then $0 < V_{it\omega}(1|0)-V_{it\omega}(0|0) \leqslant q_1p_1^*(1+\eta)+q_2p_2^*(1+(1-q_3)\eta\lambda+q_3\eta)-p_t(\omega)(1+q_1\eta+q_2q_3\eta+q_3\eta\lambda+q_2(q_2+q_1)\eta\lambda) = V_{it\omega}(x_i+1|x_i))-V_{it\omega}(x_i|x_i)$

Thus, $V_{it\omega}(1|0) > V_{it\omega}(0|0) \Longrightarrow V_{it\omega}(x_i+1|x_i)) > V_{it\omega}(x_i|x_i)$ if $p_t(\omega) \geqslant p_2^*(1+x_i+z_{i(t-1)})$.

(2). If $p_t(\omega) \leqslant p_2^*+z_{i(t-1)}(p_2^*-p_1^*) \leqslant p_2^*$,
then $V_{it\omega}(1|0) > V_{it\omega}(0|0) \Longleftrightarrow p_t(\omega) < \frac{(q_1p_1^*+q_2p_2^*)(1+\eta)+q_1q_2\eta(\lambda-1)z_{i(t-1)}(p_1^*-p_2^*)}{1+(q_1+q_2)\eta+q_3\eta\lambda}$. $(****)$
Now consider $V_{it\omega}(x_i+1|x_i), V_{it\omega}(x_i|x_i)$.

(i) If still, $p_t(\omega) \leqslant p_2^*+(x_i+z_{i(t-1)})(p_2^*-p_1^*)$, then
$V_{it\omega}(x_i+1|x_i)) > V_{it\omega}(x_i|x_i) \Longleftrightarrow p_t(\omega) < \frac{(q_1p_1^*+q_2p_2^*)(1+\eta)+q_1q_2\eta(\lambda-1)(z_{i(t-1)}+x_i)(p_1^*-p_2^*)}{1+(q_1+q_2)\eta+q_3\eta\lambda}$.
The latter inequality directly comes from $(****)$.

(ii) If $p_2^*+(x_i+z_{i(t-1)})(p_2^*-p_1^*) < p_t(\omega) \leqslant p_2^*+z_{i(t-1)}(p_2^*-p_1^*)$, then $0 \leqslant \mu((z_{i(t-1)}+1)p_2^*-p_t(\omega)-z_{i(t-1)}p_1^*) = \eta((z_{i(t-1)}+1)p_2^*-p_t(\omega)-z_{i(t-1)}p_1^*) \leqslant \eta\lambda((z_{i(t-1)}+1)p_2^*-p_t(\omega)-z_{i(t-1)}p_1^*)$.
Then $0 < V_{it\omega}(1|0)-V_{it\omega}(0|0) \leqslant q_1p_1^*(1+\eta)+q_2p_2^*(1+(1-q_1)\eta+q_1\eta\lambda)-p_t(\omega)(1+q_1\eta+q_1q_2\eta\lambda+q_3\eta\lambda+q_2(q_2+q_3)\eta) = V_{it\omega}(x_i+1|x_i))-V_{it\omega}(x_i|x_i)$

Thus, $V_{it\omega}(1|0) > V_{it\omega}(0|0) \Longrightarrow V_{it\omega}(x_i+1|x_i)) > V_{it\omega}(x_i|x_i)$ if $p_t(\omega) \leqslant p_2^*+z_{i(t-1)}(p_2^*-p_1^*)$.

(3). If $p_2^*+z_{i(t-1)}(p_2^*-p_1^*) < p_t(\omega) \leqslant p_2^*$, then directly $p_2^*+(z_{i(t-1)}+x_i)(p_2^*-p_1^*) < p_t(\omega) \leqslant p_2^*$.
Thus $V_{it\omega}(1|0)-V_{it\omega}(0|0) = V_{it\omega}(x_i+1|x_i)-V_{it\omega}(x_i|x_i) = q_1p_1^*(1+\eta)+q_2p_2^*(1+q_1\eta\lambda+(1-q_1)\eta)-p_t(\omega)(1+q_1\eta+q_2(q_2+q_3)\eta+q_3\eta\lambda+q_1q_2\eta\lambda)$
That is, $V_{it\omega}(1|0) > V_{it\omega}(0|0) \Longrightarrow V_{it\omega}(x_i+1|x_i)) > V_{it\omega}(x_i|x_i)$

(4). If $p_2^* \leqslant p_t(\omega) < p_2^*(1+z_{i(t-1)})$, then directly $p_2^* \leqslant p_t(\omega) < p_2^*(1+x_i+z_{i(t-1)})$.
Thus $V_{it\omega}(1|0)-V_{it\omega}(0|0) = V_{it\omega}(x_i+1|x_i)-V_{it\omega}(x_i|x_i) = q_1p_1^*(1+\eta)+q_2p_2^*(1+(1-q_3)\eta\lambda+q_3\eta)-p_t(\omega)(1+q_1\eta+q_2q_3\eta+q_3\eta\lambda+q_2(q_1+q_2)\eta\lambda)$
That is, $V_{it\omega}(1|0) > V_{it\omega}(0|0) \Longrightarrow V_{it\omega}(x_i+1|x_i)) > V_{it\omega}(x_i|x_i)$
This completes the proof.

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