Bayesian Persuasion in Sequential Tullock Contests*

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Abstract

We study optimal information disclosure through Bayesian persuasion in a two-player sequential contest. The winning value of the leading contestant is commonly known by both contestants and the contest organizer, while the valuation of the following contestant is his private information. The contest organizer can pre-commit to a signal to influence the leader's belief about the follower. We show that even if the distribution of the following contestant valuation is as simple as binary distributions, Bayesian persuasion can generate higher total effort than either no disclosure or full disclosure. This happens when the winning valuation of leading contestant is significantly greater than following contestant while the prior probability that following contestant has relatively high value is below some threshold. We also study the incentive yo share information for leading contestant. The results provide insights for settings where two parties are competing sequentially with one-sided asymmetric information, such as litigation.

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1 Introduction

Contests are everywhere in our lives. Examples of contests include lobbying and election campaigns among politicians, patent races among firms as well as promotion tournaments in organizations. In such contests, agents expend irreversible resources to win a reward. The objective of the contest organizer is to maximize the expected total effort of contestants. In a sales contest, sales manager aims at maximize total sales. In a patent race, the industry organizer hopes to maximize total investment on some specific technology.

As is well documented in the literature, agents (or called contestants) may have private information about their own capabilities or the valuation of the prize, etc. This naturally raises the question of information disclosure in the environment of contests. The contest designer may apply different information disclosure policies to influence the beliefs of contestants about each other's type. For example, the organizer of a sports tournament might disclose players' information that would be too costly or impossible to reveal by the players themselves to achieve higher performance. Similarly, scholars compete for grants by submitting research proposals, and presumably they do not know who they are competing against unless the grant-awarding entity, whose purpose is to stimulate research, publicly discloses the list of applicants. The informativeness of rival's type affects a contestant's effort. An ordinary researcher aware of competing for a grant against a leading scholar might give up and exert little effort.

Compared with sequential contests, the simultanuous contest have gained greater attention from researchers when analyzing contest. In a survey for rent-seeking, Nitzan (1994) summarizes that models of simultaneous rent-seeking expenditures have become standard in the rent-seeking framework. Nonetheless, in many settings, effort expenditures by agents occur sequentially rather than simultaneously. In a lawsuit, the presentation of evidence occurs sequentially. Morgan (2003) finds out that the party of the sitting president always has its convention closer to the general election from 1948 to 2000 for presidential conventions in United States. In some Olympic sports such as weightlifting, diving, ski jumping and speed skating, athletes have already learned the performance of the finished players before they enter the game.

In this paper, we study how to design the optimal information disclosure strategy in a sequentiall contest environment where some players have limited information about others valuation of the prize. We allow the information disclosure policy to take the stochastic approach of Bayesian persuasion proposed by Kamenica and Gentzkow (2011), which is a generalization of the traditional discrete information disclosure policy. The Bayesian persuasion framework has requirements for commitment power, and this assumption is generally satisfied in contests because of reputation concerns of the contest designer.

The literature on information disclosure in contests motivates our research. As mentioned earlier, the majority of researches are worked out under static contests. Some of these studies are conducted under the framework of the Tullock contests (Fu, Jiao and Lu, 2011, Hurley and Shogren, 1998*a*,*b*, Serena, 2016, Warneryd, 2003, Wasser, 2013), while the others focus on all-pay auction framework (Cai et al., 2019, Chen, 2017, Fu, Jiao and Lu, 2014, Lu, Ma and Wang, 2018, Morath and Munster, 2008). Information disclosure in simultaneous contests is well studied, while, to the best of our knowledge, there is lack of research about the optimal design of information disclosure in sequential contests. What's more, majority of existing studies on information disclosure only consider two kinds of strategy, no disclosure and full disclosure. With these two strategies, the uninformed player's beliefs either remains the same as his prior (in the no disclosure case) or coincides with the realized state (in the full disclosure case), before a contest starts. This zero-or-one choice seems straightforward. However, this restriction of choice causes loss of generality from the organizer's perspective because the organizer in real life can often choose some disclosure policies in between to partially reveal the information.

Our paper also belongs to the literature on sequential contests. Simultaneous Tullock contest have been well explored in the literature since the book written by Buchanan, Tollison and Tullock (1980). However, there are much fewer studies regarding the sequential version of this type of contests (Fu, 2006, Leininger, 1993, Leininger and Yang, 1994, Linster, 1993, Morgan, 2003, Shogren and Baik, 1992). Leininger (1993) is among the first to compare sequential-move setup with simultaneous-move setup in two-player complete-information Tullock contests. By allowing each player to choose the timing of his move (early move versus late move), he shows that in the case of asymmetric valuations, there is a unique equilibrium where the weaker player moves first and the strong player moves second, and that the total effort under this equilibrium is lower than under simultaneous play. Linster (1993) conducts a similar study without endogenizing the timing decision, compares the total effort levels among two sequential-move cases and the simultaneous-move case, as well as compares a player's payoff between moving first and simultaneous move. By assuming that players' valuations on the object follow a common binary distribution and that valuations are not realized until the order of movies is determined, Morgan (2003) conducts welfare comparisons between sequential move and simultaneous move. Chen, Kuang and Zheng (2017) studies the optimal joint design of timing and information disclosure in a 2-player private-information Tullock contest. Our work differs from the above by considering the probabilistic disclosure policies, which may potentially generate a higher total effort level.

The Bayesian persuasion approach pioneered by Kamenica and Gentzkow (2011) provides us with new methodology to handle this problem. For the contest models, Zhang and Zhou (2016), which is closely related to this article, first applies this Bayesian persuasion approach to study optimal information disclosure in 2-player simultaneous Tullock contests, and they found that when the type of informed contestant is binary, it suffices to only compare naive strategies. Other related works including Chen, Kuang and Zheng (2018) and Kuang (2019) on simultaneous and sequentiall all-pay contest and Kuang, Zhao and Zheng (2019) on simultaneous all-pay contest with two sided information asymmetry. In this article, we adopt the Bayesian persuasion approach to study the optimal information disclosure problem in sequential Tullock contests, and show that even with distributions as simple as Bernoulli distributions, Bayesian persuasion can generate higher total effort than naive strategies.

In our sequential contest model, there are two contestants, the leading contestant A, with commonly known valuation v_A and the following contestant B with privately known valuation. Player B's valuation about the prize is private information and are drawn from a commonly known binary distribution. Player B's type can be ex-post heterogeneous, which takes the value of either high or low. The most important feature of this contest model is that the contest designer can pre-commit to a signal before the contest starts. This signal can be seen as a conditional distribution on contestant B's valuation. After receiving the signal realization, both agents update their belief simultaneously to reach a common posterior. Finally, both agents engage in a sequential contest by choosing their efforts one-by-one.

Our paper has three main contributions. Firstly, we prove the uniqueness of perfect Bayesian equilibrium for 2-player sequential Tullock contests with incomplete information, which is generally note the case for sequential contests with more than 2 agents. This avoids the equilibria selection problem of this study. Secondly, we strengthen the result of Zhang and Zhou (2016) that the optimal Bayesian persuasion is degenerated to full disclosure or full concealment for simultaneous contests in the binary type case. Zhang and Zhou (2016) mainly focus on the interior solution and we show that their finding still holds when allowing for boundary solutions. Last but not least, we characterize the optimal information disclosure strategy via Bayesian persuasion for sequential contests, and we show that even with binary type distributions, Bayesian persuasion can potentially generate higher total effort than naive strategies.

The analysis of optimal Bayesian persuasion signal provides the main result of this paper. The result how Bayesian persuasion improves the organizer's payoff is similar between simultaneous contests and sequential contests to some extent. Contest designer will adopt full concealment policy when the winning valuation of contestant A is less than some threshold, which is the square root of the product of contestant B's possible valuations under simultanoues contest (Zhang and Zhou, 2016) and two times such square root under sequential contest. As the winning valuation of contestant A increasing, contestant A has valuation "closer" to stronger B than weaker B. Threshold is doubled under sequential contest demonstrates that such information is more valuable to contest designer under sequential contest and contest designer gains less by letting contestant A compete severely with stronger (has higher valuation) B.

However, when the A's valuation exceeds the threshold, result is completely different between simultaneous and sequential contests, which implies the order of the contest does matter. We show that even the contestant B's valuation follows a binary distribution, it is loss of generality to focus on full disclosure or full concealment in a sequential setup, which contradicts to the result in simultaneous contest (Zhang and Zhou, 2016). For simultaneous contests, full disclosure will always be the optimal choice, while for sequential contests, full disclosure is optimal only when valuation of A is not too much greater than such threshold.

When the valuation of contestant A is much greater than the higher value of contestant B, namely higher than two times the winning valuation of higher value, Bayesian persuasion can potentially do better than full disclosure and full concealment. In such situation, under full disclosure policy, contestant A will win the game by submitting the realized winning value of contestant B, to stop contestant B from entering the competition. Insightfully, the contest designer can distort the information under full disclosure by disguising some of the weak type contestant B as the strong type one such that contestant A will still exert higher value after Bayesian updating. Thus the probability that contestant A is exerting higher effort (when facing a strong type contestant B) increases, and therefore the contest designer can achieve higher expected total effort. This fact creates an opportunity for Bayesian persuasion to generate higher total effort than full disclosure or full concealment.

When contestant B's valuation takes more than (including) three different values, the analysis becomes more complex and a closed form solution seems unavailable. Zhang and Zhou (2016) provide a method of simplification for their model for simultaneous contests when some edges are convex in their setup, but this method no longer works in our sequential setup because all edges are strictly concave.

For the extension of our model, we study the incentives of contestants to share private information under such sequential Tullock contests framework. Wu and Zheng (2017) and Kovenock, Morath and Munster (2015) studies the incentives to share private information ahead of contests under the Tullock contest framework and the all-pay auction contest framework, respectively. Under the asymmetric information sequential contest framework established in this paper, we analyze the incentive to share value information for following contestant B. We show that there exists a unique threshold for winning value of leading contestant such that contestant B chooses to reveal his type if and only if leading contestant has winning value lower than this threshold. This incentive is unrelated to posterior distribution under binary distribution. It is worth noting that contest designer strictly prefer full concealment in such situation which indicates the conflict of interest between the contest organizer and contestant B. What's more, when introducing incentives to share information, the optimal Bayesian persuasion signal we solved in benchmark model remains optimal.

The rest of the paper is organized as follows. In Section 2, we provide an example that Bayesian persuasion improves the expected total effort for contest designer with Bernoulli distribution. In Section 3, we describe the model. In Section 4, we characterize the equilibrium in the posterior contest game and prove the uniqueness of equilibrium. In Section 5, we solve the optimal signal for the sequential contests of binary types. In Section 6, we strengthen the result of Corollary 1 in Zhang and Zhou (2016) by allowing boundary solutions. In Section 7, we provide the analysis of incentive for information sharing from the perspective of contestant. In Section 8, we conclude. Technical proofs are relegated to the appendix.

2 An Example

We use the following example to preview the main results of this article. In this section, we assume that the winning valuation of contestant A, $v_A = 4.8$. The common prior belief of v_B is

$$v_B = \begin{cases} 1 & \text{w.p.} \quad 0.5 \\ 2 & \text{w.p.} \quad 0.5 \end{cases}$$

Under the full concealment policy, the posterior distribution remains the prior. The expected total effort for contest designer is $\frac{1+\sqrt{2}}{2} \approx 1.2071$. Under the full disclosure policy, the posterior distribution is either pure distribution with $v_B = 1$ (with half probability) or pure distribution with $v_B = 2$ (with half probability). The expected total effort for contest designer is 1.5. Kamenica and Gentzkow (2011) tells us that some distribution of posteriors τ is attainable if and only if τ satisfying Bayes plausible condition. The following example illustrates what is a signal numerically.

Example 2.1 (Signal)**.** *Assume that the prior belief about v_B is*

$$v_B = \begin{cases} 1 & w.p. \quad 0.5 \\ 2 & w.p. \quad 0.5 \end{cases}$$

and the designer pre-commits to the following signal π with the realized type space $\{h, l\}$,

$$\begin{aligned} \Pr(\pi = h|v_B = 1) &= 0.2 \qquad \Pr(\pi = l|v_B = 1) = 0.8 \\ \Pr(\pi = h|v_B = 2) &= 1.0 \qquad \Pr(\pi = l|v_B = 2) = 0.0 \end{aligned}$$

After observing the realization of π, contestant A can update his belief by Bayes' rule as

$$\begin{aligned} \Pr(\pi = h) &= \frac{3}{5} \qquad \Pr(\pi = l) = \frac{2}{5} \\ \Pr(v_B = 1|\pi = h) &= \frac{1}{6} \qquad \Pr(v_B = 2|\pi = h) = \frac{5}{6} \\ \Pr(v_B = 1|\pi = l) &= 1 \qquad \Pr(v_B = 2|\pi = l) = 0 \end{aligned}$$

Signal shown in Example 2.1 is Bayes plausible because the aggregate probability distribution of v_B is still half and half. However, such distribution of posteriors cannot be generated by full concealment or full disclosure, which implies that Bayesian persuasion framework indeed generalizes the information disclosure policy.

In Figure 1, we use blue to denote the low type of contestant B and red to denote the high type of contestant B. The lowest panel represents no disclosure policy. Posterior distribution remains at prior and purple is used as fill color because it is a mixture of blue and red. The middle panel represents full disclosure policy. We use blue rectangle representing the posterior associated with v_L and red rectangle representing the posterior associated with v_H. The highest panel of Figure 1

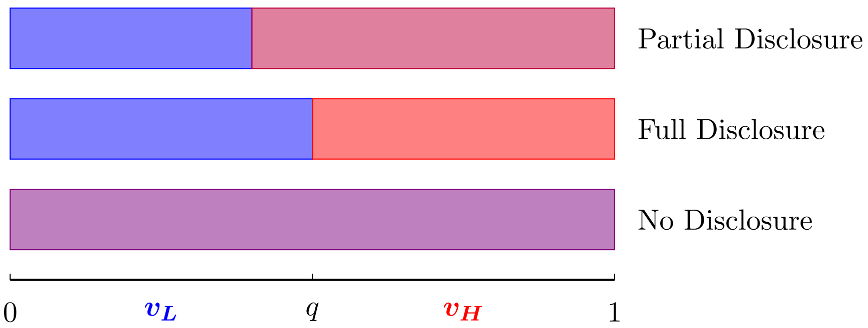

Figure 1: Three Available Bayesian Persuasion Strategies

shows posterior distributions generated by signal in Example 2.1. We use the blue rectangle to represent the posterior when receiving signal l because it is still a degenerated distribution with full probability on v_L. However, the probability of v_H is less than 1 in posterior distribution when receiving signal h, as it is shown by rasberry rectangle.

If contest designer adopts Bayesian persuasion by applying signal in Example 2.1, the expected total effort raises to 1.6, higher than total effort level in both full concealment scenario and full disclosure scenario. We call this a partial disclosure policy because only part of the incomplete information has been resolved. What's more, the signal in Example 2.1 can be shown to be the optimal signal among all possible signals. We show that although the state of the informed contestant (the following contestant) is as simple as binary, it may be payoff-improving for the designer to apply Bayesian persuasion. More specifically, Bayesian persuasion is advantageous if and only if contestant A's valuation v_A is greater than $2v_H$, where v_H denote the higher possible valuation of contestant B, and q, the probability that the type of contestant B is low, is larger than $1 - \frac{2v_H}{v_A}$.

As stated in the introduction, with extremely higher valuation, leading contestant A has incentive to exert effort level equal to winning valuation of contestant B to win deterministic in sequential contest with complete information. Furthermore, compared with full disclosure, the contest designer can distort the information by disguising some of the weak type contestant B as the strong type such that contestant A will still exert higher value v_H after Bayesian updating. Thus the probability that contestant A is exerting higher effort (when facing a strong type contestant B) increases, and therefore the contest designer can achieve higher expected total effort. In middle panel of Figure 1, contest designer receive v_H in red area and v_L in blue area. In highest panel of Figure 1, contest designer receive v_H in rasberry area and v_L in blue area. The probability of receiving v_H increases. But this logic does not apply to the simple contest because it is impossible for contestant B to bid zero in simultaneous contest when information is complete.

3 The Model

Consider the following sequential Tullock contest with incomplete information. The sequential Tullock contest framework in our paper follows that in Linster (1993). Morgan (2003) studies symmetric sequential Tullock with incomplete information, while in our study we focus on asymmetric settings. Formally, two risk-neutral contestants, A and B, compete for a single prize by exerting irreversible efforts sequentially. Contestant A moves first and contestant B moves afterwards. The success function of contestant $i \in \{A, B\}$ under effort portfolio (x_A, x_B) is given by

$$s_i(x_A, x_B) = \frac{x_i}{x_A + x_B} \tag{1}$$

If both exert zero effort, prize is assigned to contestant B to avoid trivial case because contestant B can exert infinitesimal $\varepsilon > 0$ to win the game. A contestant's payoff has a linear form, with his valuation v_i of winning multiplied by the winning probability p_i minus the cost of effort.

$$\Pi_i = s_i v_i - x_i \tag{2}$$

Contest designer tries to maximize the expected total effort Π,

$$\Pi = x_A + x_B \tag{3}$$

Contestant A's valuation of winning is commonly known as v_A. Contestant B's value of winning v_B is his private information with common prior about it shared by the contest designer and contestant A. To be more specific, v_B is a discrete random variable on Ω with $N \geq 2$ values $v_1 < \cdots < v_N$. Let $\Delta^{N-1} = \{\boldsymbol{p} \in \mathbb{R}^N : p_n \geq 0, \sum_{n=1}^{N} p_n = 1\}$ denote the standard $(N-1)$-simplex in $\mathbb{R}^N$ and $\text{int}(\Delta^{N-1})$ denote the interior of Δ^{N-1}. For each point $\boldsymbol{p} \in \text{int}(\Delta^{N-1})$, the probability for each valuation is strictly positive. Denote the prior distribution of v_B as $\bar{\boldsymbol{p}}$. Assume that $\bar{\boldsymbol{p}} \in \text{int}(\Delta^{N-1})$. Since total effort function depends on posterior distribution $\boldsymbol{p}$, we use $\Pi(\boldsymbol{p})$ to denote the total effort under $\boldsymbol{p}$, which slightly abuse the notation.

Bayesian persuasion is studied extensively after Kamenica and Gentzkow (2011). Following the spirit of Bayesian persuasion and the previous study on simultaneous Tullock contests by Zhang and Zhou (2016), we allow the contest designer to pre-commit to a signal before the contest starts in order to maximize expected total effort. A signal π consists of a realization space S and a family of likelihood distributions $\pi = \{\pi(\cdot|v_n)\}_{n=1}^{N}$ over S. For each v_n, the signal generates a distribution over the signal space S, and the signal π can be represented by an $N \times |S|$ matrix. Potential instruments for the contest designer are quite rich, including the no disclosure and full disclosure policies.

When a signal $s \in S$ is realized, contestant A needs to update his belief about contestant B by applying Bayes' rule. Denote this posterior belief as $\boldsymbol{p}_s \in \Delta^{N-1}$, which may lie on the boundary of Δ^{N-1}.

The timing of the game is as follows.

1. The contest designer chooses and pre-commits to a signal π.
2. Nature moves and draws a valuation for contestant B, say v_n.
3. The contestant designer carries out his commitment and a signal realization $s \in S$ is generated according to $\pi(s|v_n)$.
4. The signal realization s is observable by the public and leads to a posterior belief of contestant B, denoted as μ.
5. The contest takes place, contestant A chooses his effort, then contestant B chooses his effort afterwards.

Note that decisions are made only in stage 1 (contest designer) and stage 5 (contestants). We call stage 1 the *Bayesian persuasion stage* and stage 5 the *posterior contest game*, following the terminologies in Zhang and Zhou (2016). The posterior game is a sequential contest with incomplete information on the second mover. In the Bayesian persuasion stage, the contest designer is willing to choose the optimal signal π to maximize the expected total effort. We work backward and first examine the posterior contest game.

4 The Posterior Contest Game

In the posterior contest game, contestant A's valuation is commonly known as v_A and contestant B's valuation is commonly believed to be drawn from the distribution μ_s. The equilibrium of such a game is summarized in the following proposition.

Proposition 4.1. *In an incomplete information sequential Tullock contest with two contestants, leading contestant A and following contestant B, where A's valuation is commonly known as v_A and B's valuation is distributed according to $\mu \in \Delta^{N-1}$ on $(v_1, \cdots, v_N)$, there exists a unique perfect Bayesian equilibrium in which contestant A chooses effort x_A^* such that*

$$x_A^* \in \{\min x_A \in \mathbb{R} : \frac{\partial \Pi_A}{\partial x_A}^+ \leq 0\} \tag{4}$$

where

$$\frac{\partial \Pi_A}{\partial x_A}^+ = \sum_{n=1}^{N} \frac{\mu_n v_A}{2\sqrt{x_A v_n}} \mathbb{I}(x_A < v_n) - 1$$

and contestant B chooses effort according to

$$x_B^*(v_n) = \max(\sqrt{v_n x_A^*} - x_A^*, 0) \tag{5}$$

and the expected total effort is

$$\Upsilon(\mu) = \mathbb{E}_\mu(\max(\sqrt{v_n x_A^*}, x_A^*)) \tag{6}$$

If we follow the interior solution assumption made by Zhang and Zhou (2016), we can simplify this problem to a great extent. A sufficient condition to guarantee this is to assume $v_1 > \frac{v_A}{2}$. Note that the necessary and sufficient condition is

$$\frac{v_A^2}{4}\mathbb{E}^2[\frac{1}{\sqrt{v_B}}] < v_1 \tag{7}$$

which is much more complicated than $v_1 > \frac{v_A}{2}$.

Proposition 4.2. *With condition $v_1 > \frac{v_A}{2}$, in the unique perfect Bayesian equilibrium, contestant A chooses effort*

$$x_A^* = \frac{v_A^2}{4}\mathbb{E}^2[\frac{1}{\sqrt{v_B}}] \tag{8}$$

the expected total effort in this equilibrium is

$$\Upsilon(\mu) = \mathbb{E}[\sqrt{v_B}]\sqrt{x_A^*} = \frac{v_A}{2}\mathbb{E}[\sqrt{v_B}]\mathbb{E}[\frac{1}{\sqrt{v_B}}] \tag{9}$$

sequential Tullock contest generates higher expected total effort than simultaneous Tullock contest if and only if the harmonic mean of v_B is less than v_A, namely,

$$\mathbb{E}[\frac{1}{v_B}] > \frac{1}{v_A} \tag{10}$$

The proof is straightforward because the expected total effort in simultaneous Tullock contest, by Zhang and Zhou (2016), is

$$\frac{\mathbb{E}[\sqrt{V_B}]\mathbb{E}[\frac{1}{\sqrt{V_B}}]}{\frac{1}{v_A} + \mathbb{E}[\frac{1}{V_B}]}$$

5 Sequential Contests with Binary Types

5.1 Bayesian Persuasion

In stage 1, the contest designer chooses the signal π to maximize the expected total effort in the contest. Given a signal realization s, this leads to a posterior belief $\boldsymbol{p}_s$ and total effort $\Pi(\boldsymbol{p}_s)$. Denote a distribution of posteriors as $\tau \in \Delta(\Delta^{N-1})$. τ is a random variable that takes value in the simplex Δ^{N-1}. Namely, it assigns a probability measure on the posteriors in the support of τ, $\tau = \{(\Pr(s), \boldsymbol{p}_s)\}_{s\in S}$ where $\Pr(s) > 0$ denotes the probability observing signal s and $\sum_{s\in S}\Pr(s) = 1$. We call τ Bayesian plausible if the expected posterior probability equals the prior,

$$\sum_s \Pr(s)\boldsymbol{p}_s = \bar{\boldsymbol{p}} \tag{11}$$

Kamenica and Gentzkow (2011) show that finding optimal signal π is equivalent to searching over Bayesian-plausible distribution of posteriors τ, which maximize the expected value of the posterior expected total effort $\mathbb{E}_s(\Pi(\boldsymbol{p}_s)) = \sum_s \Pr(s)\Pi(\boldsymbol{p}_s)$. We can formally define the problem faced by contest designer as

$$\begin{aligned} \max_{\tau} \quad & \mathbb{E}_s(\Pi(\boldsymbol{p}_s)) \\ s.t. \quad & \sum_s \Pr(s)\boldsymbol{p}_s = \bar{\boldsymbol{p}} \end{aligned}$$

The indirect value function from the above maximization problem is exactly equal to the value of the concave closure of $\Pi(\boldsymbol{p})$ at the prior, denoted as $\mathbf{cav}\Pi(\bar{\boldsymbol{p}})$, as is shown in the following proposition (Kamenica and Gentzkow, 2011).

Proposition 5.1. *The optimal signal always exists and achieves an expected total effort equal to* $\mathbf{cav}\Pi(\bar{\boldsymbol{p}})$.

The optimal signals are quite simple in some special cases. When $\Pi(\boldsymbol{p})$ is concave, then no disclosure is optimal. When $\Pi(\boldsymbol{p})$ is convex, then full disclosure is optimal. In order to find the optimal signal, we need to construct the concave disclosure of $\Pi(\boldsymbol{p})$ on the simplex Δ^{N-1}. In this paper, we pay most of our attention to the binary case as the result is quite positive even in such simple circumstances.

When contestant B's valuation follows a binary distribution, let $v_1 = v_L$ and $v_2 = v_H$. In the posterior contest game, the belief $\boldsymbol{p} = (q, 1-q)$, where q denotes the probability of low valuation. We use q instead of p to avoid confusion. We further use $\phi(q)$ to represent the expected total effort function depending on q to avoid confusion. Note that although the prior is interior, the posterior belief may lie on the boundary, $q \in [0, 1]$.

5.2 Sequential Contests

Compared with simultaneous contests, the analysis becomes more complicated because we need to consider 4 different scenarios according to the numerical value of v_A, v_H, v_L, while there are only 2 cases to consider in simultaneous contests (the interior and boundary solution cases, refer to Section 6). To begin with, let us summarize the perfect Bayesian equilibrium of such a contest. Assume the posterior $q \in [0, 1]$ and in the perfect Bayesian equilibrium, contestant A exerts x_A^*, low type contestant B exerts x_{BL}^* and high type contestant B exerts x_{BH}^*.

Proposition 5.2. *Contestant A has four different potential strategies.*

1. $x_A < v_L$ *such that both type of contestant B will join the contest;*

2. $x_A = v_L$ *to push the low type contestant B out of the competition but more effort is dominated;*

3. $x_A \in (v_L, v_H)$*;*

4. $x_A = v_H$ *to push both types of contestant B out of the competition.*

Any $x_A > v_H$ *is dominated by strategy 4.*

A brief analysis of strategy *1* and strategy *3* are shown in the following lemmas.

Lemma 5.3. *When exerting* $x_A \leq v_L$*, the expected utility for contestant A will be*

$$\Pi_A = q\frac{x_A}{\sqrt{x_A v_L}}v_A + (1-q)\frac{x_A}{\sqrt{x_A v_H}}v_A - x_A \tag{12}$$

with first order condition

$$\frac{\partial \Pi_A}{\partial x_A} = \frac{v_A}{2\sqrt{x_A}}\Big(\frac{q}{\sqrt{v_L}} + \frac{1-q}{\sqrt{v_H}}\Big) - 1 \tag{13}$$

and second order condition

$$\frac{\partial^2 \Pi_A}{\partial x_A^2} < 0$$

Lemma 5.4. *When exerting* $x_A \in [v_L, v_H]$*, the expected utility for contestant A will be*

$$\Pi_A = v_A + (1-q)\frac{x_A}{\sqrt{x_A v_H}}v_A - x_A \tag{14}$$

with first order condition

$$\frac{\partial \Pi_A}{\partial x_A} = \frac{v_A}{2\sqrt{x_A}}\Big(\frac{1-q}{\sqrt{v_H}}\Big) - 1 \tag{15}$$

and second order condition

$$\frac{\partial^2 \Pi_A}{\partial x_A^2} < 0$$

By comparing the outcomes of these four strategies, we summarize the equilibrium of such a sequential contest in the following theorem.

Theorem 5.5 (Perfect Bayesian Equilibrium)**.** *In the perfect Bayesian equilibrium, the optimal strategy of contestant A is*

$$x_A = \begin{cases} \frac{v_A^2}{4}(\frac{q}{\sqrt{v_L}} + \frac{1-q}{\sqrt{v_H}})^2 & \textbf{\textit{if}}\ v_A < 2v_L\ \textbf{\textit{or}}\ v_A \in [2v_L, 2\sqrt{v_H v_L})\ \textit{and}\ q \in [0, q_1) \\ v_L & \textbf{\textit{if}}\ v_A \in [2v_L, 2\sqrt{v_H v_L})\ \textit{and}\ q \in [q_1, 1]\ \textbf{\textit{or}}\ v_A \geq 2\sqrt{v_H v_L}\ \textit{and}\ q \in [q_2, 1] \\ \frac{v_A^2}{4}(\frac{1-q}{\sqrt{v_H}})^2 & \textbf{\textit{if}}\ v_A \in [2\sqrt{v_H v_L}, 2v_H)\ \textit{and}\ q \in [0, q_2)\ \textbf{\textit{or}}\ v_A \geq 2v_H\ \textit{and}\ q \in (q_3, q_2) \\ v_H & \textbf{\textit{if}}\ v_A \geq 2v_H\ \textit{and}\ q \in [0, q_3] \end{cases} \tag{16}$$

where $q_1 = \frac{\sqrt{v_L}(2\sqrt{v_H v_L} - v_A)}{(\sqrt{v_H} - \sqrt{v_L})v_A}$, $q_2 = 1 - \frac{2\sqrt{v_H v_L}}{v_A}$ *and* $q_3 = 1 - \frac{2v_H}{v_A}$.

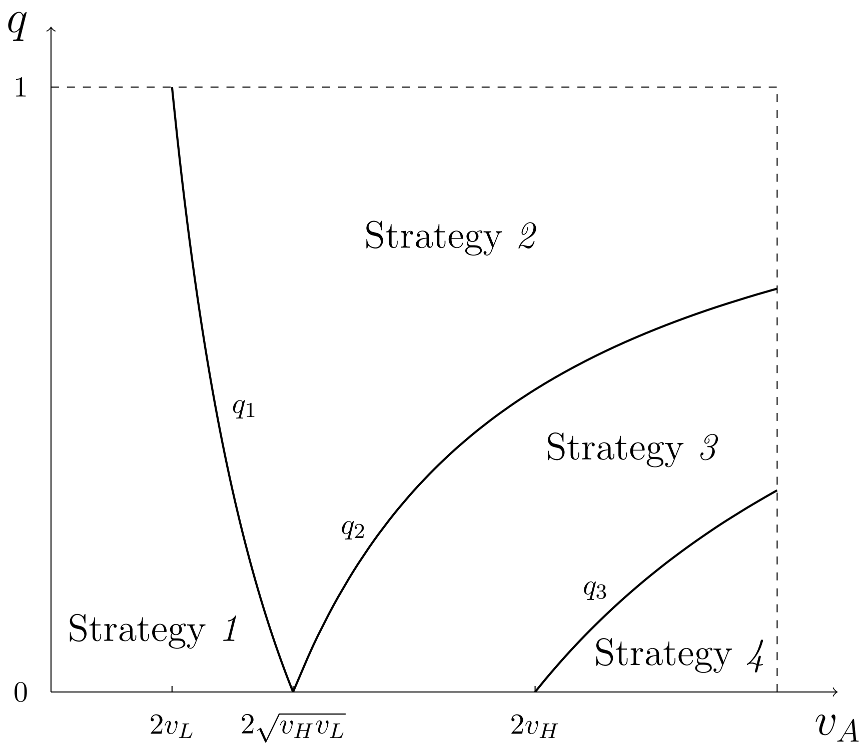

Figure 2: Four Potential Strategies

Theorem 5.5 is a little bit messy because the definition of domains depends on four parameters, v_A, v_H, v_L, q. Nonetheless, if we fix the value of v_H, v_L, q, the optimal strategy of contestant A is a four-piecewise function with two horizontal segments (Linster, 1993). When the winning value of contestant A is relatively low, the equilibrium effort increases as v_A increases. However, when v_A reaches some level, the equilibrium effort of contestant A will remain the same until v_A reaches a higher level. After that there is another increasing segment. If the winning value of contestant A is high enough, then it is profitable for contestant A to push both types of contestant B out of the competition and win the contest definitely. Figure 2 provides a graphical illustration of four potential strategies in Theorem 5.5.

Theorem 5.5 divides the whole domain into four different parts according to the magnitude of v_A given v_H and v_L. In the following subsections, we analyze the optimal Bayesian persuasion strategy with different ranges of v_A: (1) $v_A < 2v_L$; (2) $v_A \in [2v_L, 2\sqrt{v_H v_L}]$; (3) $v_A \in (2\sqrt{v_H v_L}, 2v_H]$ and (4) $v_A > 2v_H$, respectively. These classification is a little bit different from Theorem 5.5, but only in boundaries, which will not influence the result which applies concavification. We summarize the optimal Bayesian persuasion in the following theorem.

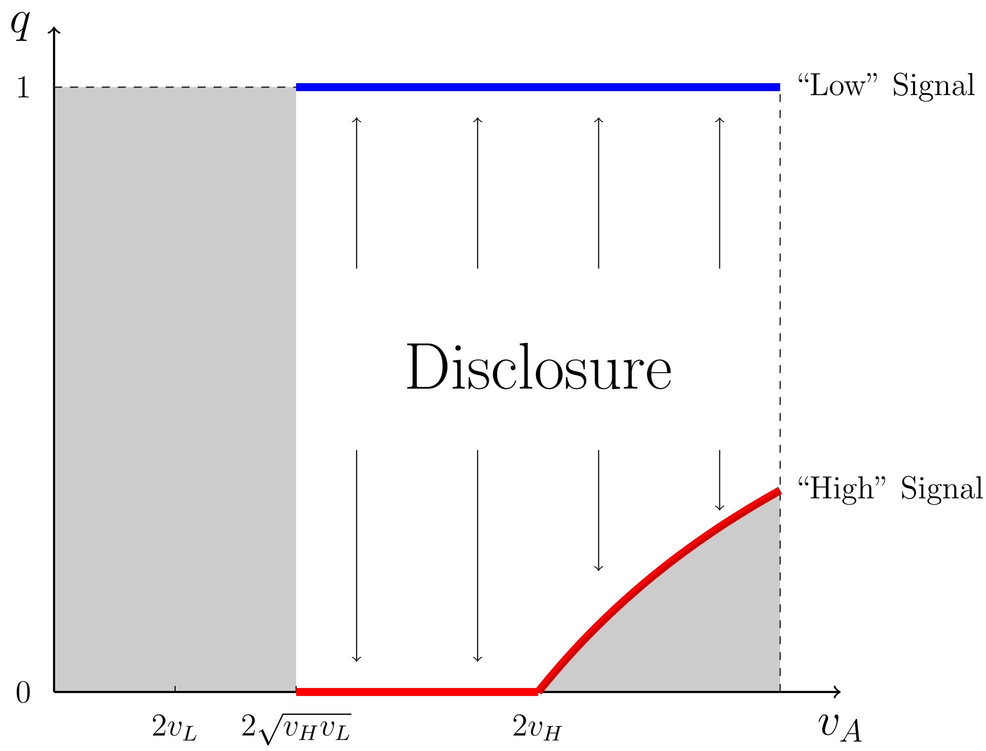

Figure 3: Optimal Bayesian Persuasion

Theorem 5.6. *The optimal Bayesian persuasion is shown in the following table.*

range of v_A	q	*optimal Bayesian persuasion*
$v_A < 2v_L$	$[0, 1]$	*no disclosure*
$v_A \in [2v_L, 2\sqrt{v_H v_L}]$	$[0, 1]$	*no disclosure*
$v_A \in (2\sqrt{v_H v_L}, 2v_H]$	$[0, 1]$	*full disclosure*
$v_A > 2v_H$	$q \in (1 - \frac{2v_H}{v_A}, 1)$	*Partial Disclosure with beliefs* $q_l = 1$ *and* $q_h = 1 - \frac{2v_H}{v_A}$
	$q \in [0, 1 - \frac{2v_H}{v_A}] \cup \{1\}$	*no disclosure*

The contest designer benefits from Bayesian persuasion if and only if $v_A > 2v_H$ *and* $q > 1 - \frac{2v_H}{v_A}$.

Figure 3 graphically illustrates Theorem 5.6. We use the horizontal axis to represent different values of v_A and the vertical axis to represent the probability q. The contest designer applies full concealment strategy when 2-tuple (v_A, q) locates in grey area. The contest designer delivers two possible signals in the white area, the "Low" signal and the "High" signal. When receiving "Low" signal, contestant B must has winning value v_L. On the contrary, when receiving "High" signal, contestant B has winning value v_H deterministically only if v_A does not exceed $2v_H$, otherwise, player A cannot accurately infer the type of opponent.

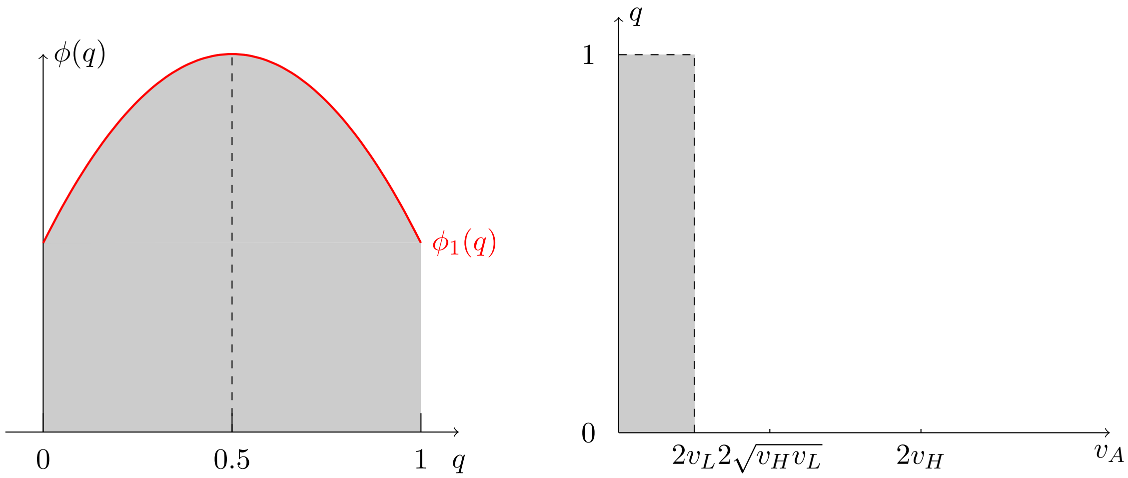

Figure 4: $v_A < 2v_L$

5.2.1 $v_A < 2v_L$

Solutions are interior since the $v_L > \frac{v_A}{2}$ condition is satisfied. The optimal strategy for contestant A is $x_A^* = \frac{v_A^2}{4}(\frac{q}{\sqrt{v_L}} + \frac{1-q}{\sqrt{v_H}})^2$ by Theorem 5.5. We can re-write the expected total effort as $\phi_1(q)$,

$$\phi(q) = \phi_1(q) = \Pi((q, 1-q)) = \frac{v_A}{2}\Big((2q^2 - 2q + 1) + (q - q^2)(\frac{\sqrt{v_H}}{\sqrt{v_L}} + \frac{\sqrt{v_L}}{\sqrt{v_H}})\Big) \tag{17}$$

Since the second order derivative $\phi_1''(q) = v_A(2 - \frac{\sqrt{v_H}}{\sqrt{v_L}} - \frac{\sqrt{v_L}}{\sqrt{v_H}}) < 0$, $\phi(q)$ is strictly concave when $q \in [0, 1]$, the concave closure of $\phi_1(q)$ is itself.

Proposition 5.7. *With binary distribution such that $v_A < 2v_L$, the no disclosure policy is optimal among all possible signals.*

Left panel of Figure 4 shows graphically that the concave closure of $\phi(q)$ is indeed itself.

5.2.2 $v_A \in [2v_L, 2\sqrt{v_H v_L}]$

According to Theorem 5.5, the necessary and sufficient condition of interior solution gives us

$$q < q_1 \leq 1 \tag{18}$$

When $q < q_1$, the total expected effort $\phi(q)$ is strictly concave according to Equation 17. From the first order condition we can show that the maximum value of $\phi(q)$ is attained at $\min(\frac{1}{2}, q_1)$. When $q \geq q_1$, contestant A will exert v_L, hence the total effort can be expressed as a linear function,

$$\phi_2(q) = qv_L + (1-q)\sqrt{v_H v_L} \tag{19}$$

with first and second order conditions

$$\phi_2'(q) = v_L - \sqrt{v_H v_L} < 0, \phi_2''(q) = 0 \tag{20}$$

Thus, the expected total effort is

$$\phi(q) = \begin{cases} \phi_1(q) & q \in [0, q_1) \\ \phi_2(q) & q \in [q_1, 1] \end{cases} \tag{21}$$

The optimal policy remains the same as in the previous case.

Proposition 5.8. *With binary distribution such that $v_A \in [2v_L, 2\sqrt{v_H v_L}]$, the no disclosure policy is optimal among all possible signals.*

Proof sketch of Proposition 5.8. If we extend the left side of $\phi(q)$, quadratic function $\phi_1(q)$, into the interval $[q_1, 1]$, we can observe that the linear function $\phi_2(q)$ is located below the concave $\phi_1(q)$ in the interval $q \in [q_1, 1]$, both function intersect at $q = q_1$ and

$$\phi_2(1) = v_L < \frac{v_A}{2} = \phi_1(1) \tag{22}$$

Hence, at the turning point q_1, the left-hand side derivative is obviously higher than the right-hand side derivative. In conclusion, the concave closure of $\phi(q)$ is itself. Please refer to Appendix for the analytical proof. □

Figure 5 show the concave closure graphically where left panel and middle panel correspond to the case when $q_1 < 0.5$ and $q_1 > 0.5$ respectively. When $q > q_1$, $\phi_1(q)$ presented by dashed curve is larger than $\phi_2(q)$ One interesting fact is that when $v_A = 2\sqrt{v_H v_L}$, contestant A will exert exactly the same amount of effort v_L regardless of whether contestant B is of valuation v_H or v_L, and any signal yields the same expected total effort. In a simultaneous Tullock contest, this critical value is $v_A = \sqrt{v_H v_L}$. Zhang and Zhou (2016) interprets this condition as both v_L and v_H are equally close to v_A. However, this explanation fails in the sequential Tullock contest. Recall that there are some well-known facts in two-player simultaneous contests with complete information (Dixit, 1987, Nti, 1999): (1) A contestant's equilibrium effort is strictly increasing in his opponent's valuation if he is the favorite and is strictly decreasing in his opponent's valuation if he is the underdog. (2) Strategies are complements for the favorite and substitutes for the underdog around equilibrium. However, both of the above facts are no longer valid in sequential Tullock contests.

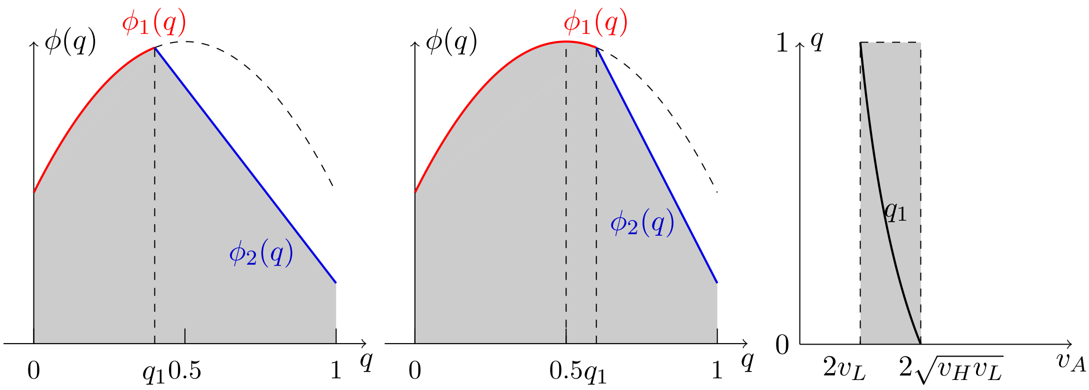

Figure 5: $v_A \in [2v_L, 2\sqrt{v_H v_L}]$

5.2.3 $v_A \in (2\sqrt{v_H v_L}, 2v_H]$

When $q < q_2$, by Theorem 5.5, the optimal strategy for contestant A is

$$x_A^* = \frac{v_A^2}{4}(\frac{1-q}{\sqrt{v_H}})^2 \tag{23}$$

Hence, we can write the total effort as a function of q,

$$\phi_3(q) = \frac{v_A^2}{4v_H}q(1-q)^2 + \frac{v_A}{2}(1-q)^2 \tag{24}$$

with first order derivative

$$\phi_3'(q) = \frac{v_A^2}{4v_H}(3q^2 - 4q + 1) + \frac{v_A}{2}(2q-2) \tag{25}$$

and second order derivative

$$\phi_3''(q) = \frac{v_A^2}{4v_H}(6q-4) + v_A \tag{26}$$

The expected total effort is

$$\phi(q) = \begin{cases} \phi_3(q) & q \in [0, q_2) \\ \phi_2(q) & q \in [q_2, 1] \end{cases} \tag{27}$$

Zhang and Zhou (2016) points out a critical value $v_A^* = \sqrt{v_H v_L}$ for a simultaneous Tullock contest such that if $v_A = v_A^*$, any signal yields the same expected total effort; when $v_A > v_A^*$, full disclosure is optimal; when $v_A < v_A^*$, no disclosure is optimal. In a sequential Tullock contest, a similar (but different) outcome is observed, with the critical value v_A^* replaced by $2\sqrt{v_H v_L}$. To be more specific, when $v_A < 2\sqrt{v_H v_L}$, no disclosure is optimal; when $v_A = 2\sqrt{v_H v_L}$, any signal is optimal. Furthermore,

Proposition 5.9. *When $v_A \in (2\sqrt{v_H v_L}, 2v_H]$, the full disclosure policy is optimal.*

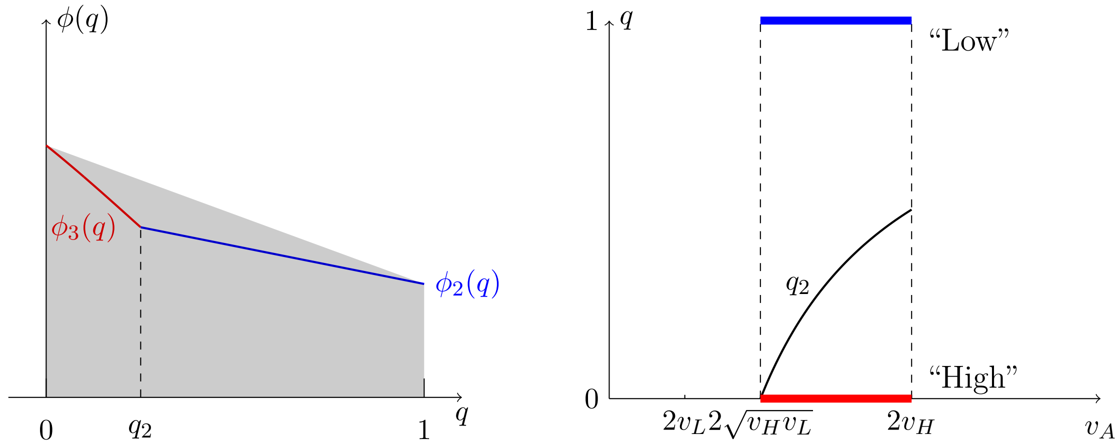

Figure 6: $v_A \in (2\sqrt{v_H v_L}, 2v_H]$

Proof sketch of Proposition 5.9. The total expected effort $\phi(q)$ is a two-piecewise function. In order to prove that full disclosure is optimal, we need to prove that for any $q \in [0,1]$, **full disclosure condition** holds. The slope of the line segment between $(0, \phi(0))$ and $(1, \phi(1))$ is

$$\phi(1) - \phi(0) = v_L - \frac{v_A}{2}$$

while when $q \in [q_2, 1]$, the derivatives $\phi_2'(q) = v_L - \sqrt{v_H v_L} > \phi(1) - \phi(0)$ and $\phi(q_2) < \phi(0) + q_2(\phi(1) - \phi(0))$. Hence, **full disclosure condition** holds when $q \in [q_2, 1]$.

Then we move to the case $q \in [0, q_2]$. If $\phi_3(q)$ is convex, **full disclosure condition** holds obviously. If $\phi_3(q)$ is concave, then we can prove that

$$\phi_3'(q) < \phi_3'(0) = \frac{v_A^2}{4v_H} - v_A < v_L - \frac{v_A}{2} = \phi(1) - \phi(0) \tag{28}$$

then **full disclosure condition** holds. If $\phi_3(q)$ is concave-convex, combine these two cases together, we can conclude that **full disclosure condition** holds. Please refer to Appendix for a complete analysis. □

Figure 6 shows the concave closure of $\phi(q)$ graphically.

5.2.4 $v_A > 2v_H$

According to Theorem 5.5, when $q \leq q_3$, contestant A will exert v_H to push contestant B out of the competition. Hence, the total effort $\phi_4(q) = v_H$ remains constant. When $q \in (q_3, q_2)$, the optimal strategy for contestant A is $x_A^* = \frac{v_A^2}{4}(\frac{1-q}{\sqrt{v_H}})^2$. The expected total effort is

$$\phi(q) = \begin{cases} \phi_4(q) & q \in [0, q_3] \\ \phi_3(q) & q \in (q_3, q_2) \\ \phi_2(q) & q \in [q_2, 1] \end{cases} \tag{29}$$

The analysis is similar to the last subsection. Since v_A is greater than $2v_H$, there may exist a non-negative posterior $q > 0$ such that v_H is still the optimal strategy for contestant A. This fact creates the possibility of the improvement from Bayesian persuasion. Let us look back to Section 2. When applying the full disclosure policy, contest designer report “HIGH” when $v_B = v_H$ and “LOW” when $v_B = v_L$. Compared with the full disclosure policy, a better solution comes by adding a non-negative probability into the high-signal. The contest designer may report “HIGH” with some probability when $v_B = v_L$ to confuse contestant A. However, this probability can be chosen such that when receiving the “HIGH” signal, contestant A will still have incentive to exert v_H. Figure 7 shows this resut graphically. Formally,

Theorem 5.10. *Under the assumption of binary distribution, the contest designer benefit from Bayesian persuasion if and only if $v_A > 2v_H$ and $q \in (q_3, 1)$. The optimal signal induces beliefs $q_h = q_3$(“HIGH”) and $q_l = 1$(“LOW”). Thus, the expected total effort after Bayesian persuasion,* $\mathbf{cav}\phi(q)$ *is a piecewise linear function,*

$$\mathbf{cav}\phi(q) = \begin{cases} v_H & q \le q_3 \\ v_L + (1-q)\frac{v_A(v_H - v_L)}{2v_H} & q > q_3 \end{cases} \tag{30}$$

Proof sketch of Theorem 5.10. The total effort $\phi_4(q) = v_H$ remains constant when $q \in [0, q_3]$. Then we want to prove that $\phi(q)$ is always located below the line segment between (q_3, v_H) and $(1, v_L)$ with slope $\frac{v_A(v_L - v_H)}{2v_H}$ in interval $q \in (q_3, 1)$. The analysis is the same as the proof to Proposition 5.9. Please refer to Appendix for details.

The optimal policy for the designer is described as follows. When $q \in [0, q_3] \cap \{1\}$, the contest designer should conceal the information; when $q \in (q_3, 1)$, the contest designer can induce extra effort by Bayesian persuasion. □

6 Bayesian Persuasion in Simultaneous Contests

In this section, we try to enhance the result shown by Corollary 1 in Zhang and Zhou (2016). First of all, we summarize the Bayesian Nash equilibrium of such a contest. Assume the posterior $q \in [0, 1]$ and in the perfect Bayesian equilibrium, contestant A exerts x_A^*, low type contestant B exerts x_{BL}^* and high type contestant B exerts x_{BH}^*.

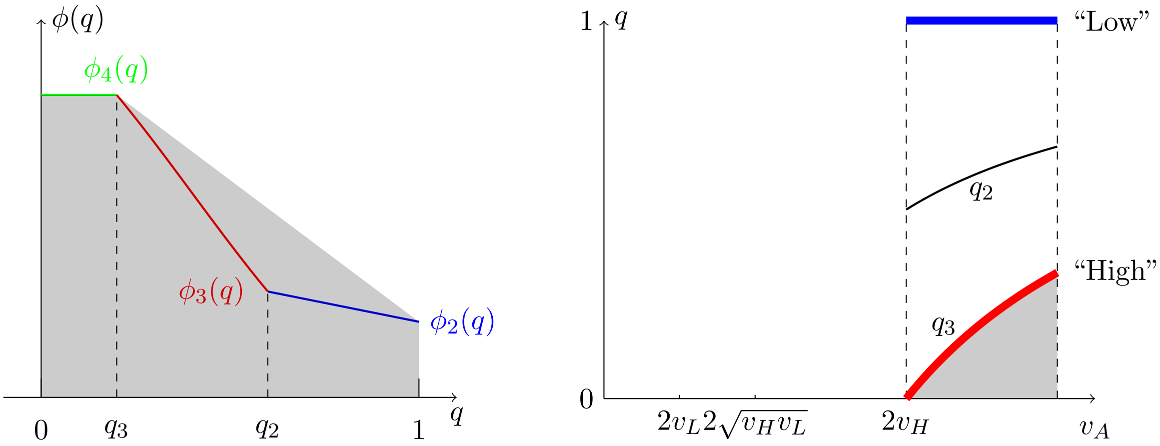

Figure 7: $v_A > 2v_H$

Lemma 6.1 (Interior Solution)**.** *Interior solution gives us*

$$x_A^* = \left(\frac{\frac{q}{\sqrt{v_L}} + \frac{1-q}{\sqrt{v_H}}}{\frac{1}{v_A} + \frac{q}{v_L} + \frac{1-q}{v_H}}\right)^2 \tag{31}$$

The solution is interior, namely $x_A^* > 0, x_{BH}^* > 0$ and $x_{BL}^* > 0$, if and only if

$$q > 1 - \frac{v_H\sqrt{v_L}}{v_A(\sqrt{v_H} - \sqrt{v_L})} \tag{32}$$

Lemma 6.2 (Boundary Solution)**.** *When $0 < q < 1 - \frac{v_H\sqrt{v_L}}{v_A(\sqrt{v_H}-\sqrt{v_L})}$, the optimal strategy for contestant A will be*

$$x_A^* = \left(\frac{\frac{1-q}{\sqrt{v_H}}}{\frac{1}{v_A} + \frac{1-q}{v_H}}\right)^2 \tag{33}$$

For the sake of simplicity, let $\phi(q) = \Pi_{\text{sim}}((q, 1-q))$ denote the expected total effort in a simultaneous Tullock contest with binary distribution. q denotes the probability of low type. Although we allow boundary solutions here, Bayesian persuasion still cannot yield higher expected total effort, which strengthens the key result in Corollary 1 of Zhang and Zhou (2016).

Theorem 6.3. *In a simultaneous Tullock contest with binary distribution, either no disclosure or full disclosure is optimal among all possible signals. In other words, Bayesian persuasion cannot yield higher effort.*

Proof sketch of Theorem 6.3. Zhang and Zhou (2016) has shown that Bayesian persuasion cannot yield higher effort when solution is interior. What we need to do is to analyze the boundary case, namely when

$$v_A > \frac{v_H\sqrt{v_L}}{\sqrt{v_H} - \sqrt{v_L}} \tag{34}$$

We claim that **full disclosure** is optimal in such cases. In other words, $\phi(q)$ lies under the line segment between $(0, \phi(0))$ and $(1, \phi(1))$. Mathematically, for any $q \in [0, 1]$

$$\phi(q) \leq \phi(0) + q(\phi(1) - \phi(0)) = (1 - q)\phi(0) + q\phi(1) \tag{35}$$

we call it **full disclosure condition**.

According to Proposition 3 in Zhang and Zhou (2016), $\phi(q)$ is strict convex inside interval $[q^*, 1]$ where $q^* = 1 - \frac{v_H\sqrt{v_L}}{v_A(\sqrt{v_H}-\sqrt{v_L})}$ because $v_A > \frac{v_H\sqrt{v_L}}{\sqrt{v_H}-\sqrt{v_L}} > \sqrt{v_H v_L}$. Hence,

$$\phi(q) < \phi(q^*) + (q - q^*)\frac{\phi(1) - \phi(q^*)}{1 - q^*} \qquad \text{when} \quad q \in [q^*, 1] \tag{36}$$

Then, we can prove by complex algebra technique that for any $q \in [0, q^*]$,

$$\phi(q) < \phi(0) + q(\phi(1) - \phi(0)) \tag{37}$$

Then, since $\phi(q^*) < \phi(0) + q^*(\phi(1) - \phi(0))$, for $q \in [q^*, 1]$,

$$\phi(q) < (1 - q)\phi(0) + q\phi(1) \tag{38}$$

Therefore, for any $q \in [0, 1] = [0, q^*] \cup [q^*, 1]$, **full disclosure condition** holds. Please refer to Appendix for the technique details. □

However, as it is shown in Section 5, Bayesian persuasion can potentially generate higher total expected effort than the full concealment or full disclosure policy in sequential Tullock contests evenwith binary distribution.

7 Extension: Contestant Can Reveal Type Information

In this section, we slightly modifies the setup in benchmark model by incorporating the incentives of contestant B to reveal type information. One more stage is included in this game, called *revealing stage*, in which contestant B chooses whether to reveal his type or not conditioning on signal realization. This stage is added immediately after *Bayesian persuasion stage*. Using backward inducntion, we first study the incentive to share information for a given posterior contest game. In this section, we only analyze two possible strategies for contestant B, revealing and concealing, without considering Bayesian persuasion.

7.1 Incentives to Share Information

We study the incentive to share information.for a given posterior distribution $(q, 1-q)$. Notice that when contestant A bid x_A, the utility function for contestant B with winning value v_B is

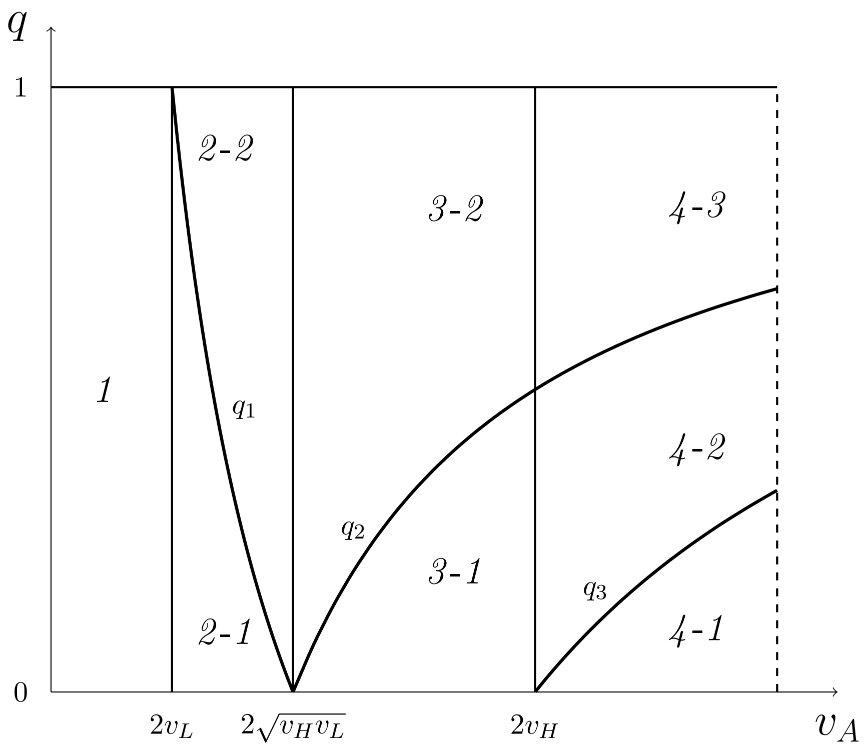

Figure 8: Eight Situations

expressed as

$$\Pi_B(x_A, v_B) = \begin{cases} (\sqrt{v_B} - \sqrt{x_A})^2 & x_A < v_B \\ 0 & x_A \geq v_B \end{cases}$$

If contestant B choose to fully share value information, the expected utility for contestant B will be

$$\begin{aligned} \Pi_B^S(q) &= \underbrace{\begin{cases} \Pi_B(\frac{v_A^2}{4v_L}, v_L) & v_A < 2v_L \\ 0 & v_A \geq 2v_L \end{cases}}_{v_A \text{ versus } v_L} \times \boldsymbol{q} + \underbrace{\begin{cases} \Pi_B(\frac{v_A^2}{4v_H}, v_H) & v_A < 2v_H \\ 0 & v_A \geq 2v_L \end{cases}}_{v_A \text{ versus } v_H} \times \boldsymbol{(1-q)} \\ &= \begin{cases} qv_L(1 - \frac{v_A}{2v_L})^2 + (1-q)v_H(1 - \frac{v_A}{2v_H})^2 & v_A < 2v_L \\ (1-q)v_H(1 - \frac{v_A}{2v_H})^2 & 2v_L \leq v_A < 2v_H \\ 0 & v_A \geq 2v_H \end{cases} \end{aligned}$$

which is always a linear function connecting $\Pi_B^S(0)$ and $\Pi_B^S(1)$.

If contestant B choose not to share value information, we have four categories according to the value of v_A and eight different situations according to Theorem 5.5, as shown in the Figure 8. The following theorem characterizes the sharing behavior of contestant B.

Theorem 7.1. *Contestant B has incentive to share information if and only if $v_A < 2\sqrt{v_H v_L}$.*

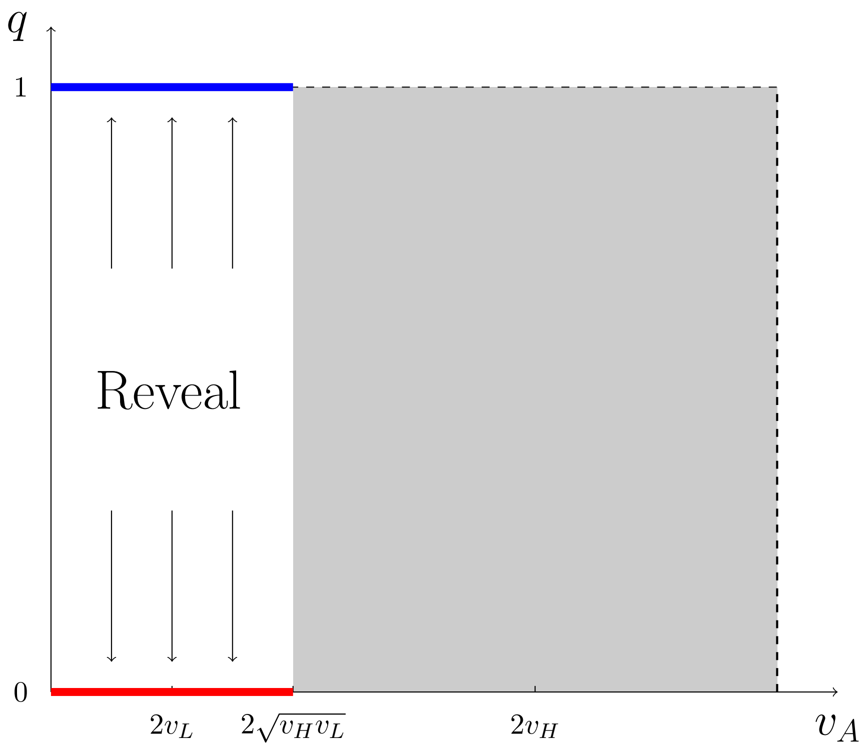

Figure 9: Incentive to share information

As it is shown in Figure 9, the strategy of contestant B is unrelated to posterior distribution. However, it is generally not true with more than two possible values. For ternary distribution with support $v_1 < v_2 < v_3$, we can set the winning value of leading contestant A, $v_A \in (2\sqrt{v_1 v_2}, 2\sqrt{v_2 v_3})$. When $\Pr(v_3) = 0$, contestant B has incentive to share type information; when $\Pr(v_1) = 0$ has no incentive to share type information.

For type information, contest designer and contestant B has severe conflict of interests. On one hand, contestant prefers concealing type information when contest designer trying to disclose such information. On the other hand, contestant prefers revealing type information when contest designer trying to conceal such information.

7.2 Bayesian Persuasion

Given the strategy of contestant B, we now go back to *Bayesian persuasion stage*. When $v_A < 2\sqrt{v_H v_L}$, no matter which signal designer applied, contestant B will reveal his type afterwards and thus forms a complete information situation. When $v_A \geq 2\sqrt{v_H v_L}$, no matter which signal designer applied, contestant B will conceal his type afterwards. This implies the following theorem.

Theorem 7.2. *The introduction of incentives for contestant B will not change the optimal Bayesian persuasion strategy for contest designer.*

Nonetheless, the posterior profile is changed because of the introduction of incentives. Through

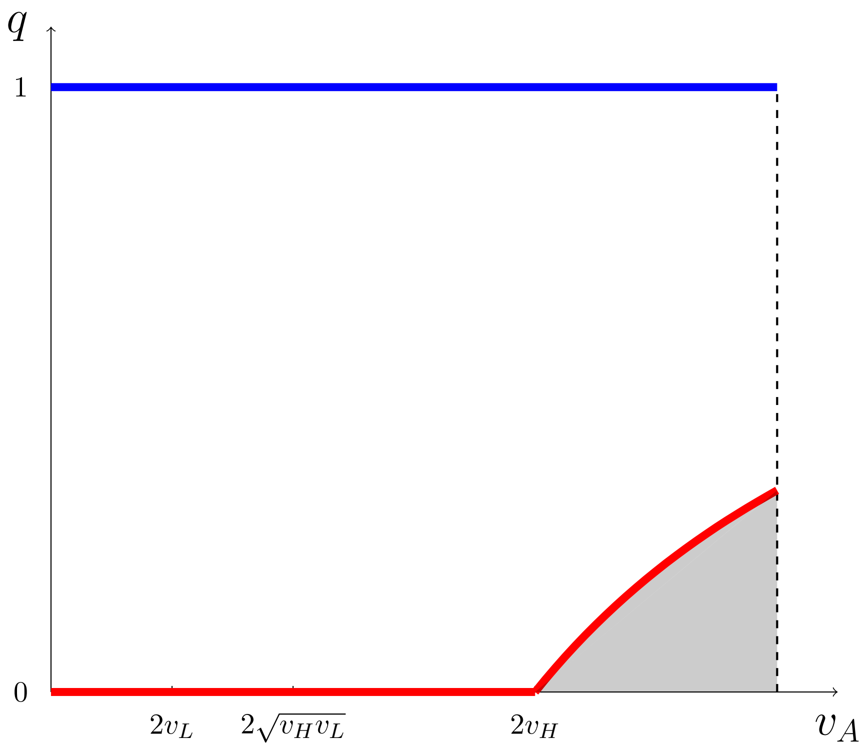

Figure 10: Posterior Profile with Incentives of Contestant B

Bayesian persuasion of contest designer and information sharing of contestant B, the final information structure can be described in Figure 10. When $v_A < 2v_H$, there are no hidden information after *revealing stage*. When $v_A > 2v_H$ and $q < 1-\frac{2v_H}{v_A}$, both contest designer and contestant B has no incentive to share type information. When $v_A > 2v_H$ and $q > 1-\frac{2v_H}{v_A}$, there are two potential posterior distributions corresponding to two signals generated by contest designer: "High" and "Low".

8 Conclusion and Discussion

In this paper, we investigate how information disclosure through the Bayesian persuasion approach can be used to enhance the expected total effort in sequential Tullock contests. In our model, the leading contestant's valuation is commonly known and the following contestant's valuation is private information. We show that even in the binary case, Bayesian persuasion may enhance the total effort compared with the no disclosure or full disclosure policy, which differs from the result for simultaneous contests in Zhang and Zhou (2016). We restrict our attention to the binary case, as solving for the equilibrium for more general cases not only raises technical challenges, but also is considered not worthwhile as Bayesian persuasion may also work for simultaneous contests with more than two types. How to compute the optimal signal in general cases is still an open question.

In Kamenica and Gentzkow (2011), their first question is when the sender could benefit from persuasion. The same question can be asked in out framework. Section 2 and Theorem 5.10 actually illustrate that contest designer can benefit from persuasion, even in the case of binary types. This happens when v_A far exceeds v_H, namely, $v_A > 2v_H$. The contest designer can report the "HIGH" signal with some probablity when $v_B = v_L$ to confuse contestant A while keeping contestant A exerting v_H after receiving the "HIGH" signal.

In simultaneous contests, we may also be interested in correlated signals when both players have private values. However, this seems very challenging even for the symmetric prior case, because we are not able to solve for Bayesian Nash equilibrium for the general case of incomplete information contests. For sequential contests, it is meaningless to consider correlated signals because contestant B only takes into account the effort of contestant A, rather than the type of contestant A.

One possible direction for future work is the timing policy of contest. Keeping the assumption of asymmetric information unchanged, the designer can organize the contest in three different timing settings: (1) simultaneous move; (2) sequential move with first mover being informed and (3) sequential move with first mover being uninformed. Contest designer can change the timing of the contest as well as to apply Bayesian persuasion before the game starts. Another direction for future work can be the scenario where the designer can choose which contestant to be persuaded, assuming both players have asymmetric distributions of types.

Appendix A: Proofs

Proof of Proposition 4.1. We focus on pure strategy equilibria. Denote the equilibrium as $(x_A^*, x_B^*(v_{Bn}))$, given contestant A's equilibrium effort x_A^*, $x_B^*(v_{Bn})$ solves

$$\max_{x_B \geq 0} \frac{x_B}{x_A^* + x_B} v_{Bn} - x_B \tag{39}$$

by first order condition, we get

$$x_B^*(v_{Bn}) = \max(\sqrt{v_{Bn} x_A^*} - x_A^*, 0) \tag{40}$$

Given contestant B's equilibrium strategy $x_B^*(v_{Bn})$, by backward induction, x_A^* solves

$$\max_{x_A \geq 0} \Pi_A = \mathbb{E}\Big[\frac{x_A}{x_A + x_B^*(v_{Bn})} v_A - x_A\Big] \tag{41}$$

rewrite this formula, we have

$$\Pi_A = \sum_{n: v_{Bn} \leq x_A} \mu_n v_A + \sum_{n: v_{Bn} > x_A} \mu_n \sqrt{\frac{x_A}{v_{Bn}}} v_A - x_A \tag{42}$$

Let v_{Bk} denote the lowest value v_{Bn} that is greater than or equal to x_A, then when $x_A \in [v_{Bk-1}, v_{Bk})$ $(v_{B0} = 0)$, the optimization problem can be written as

$$\max_{x_A \in [v_{Bk-1}, v_{Bk})} \Pi_A = \sum_{n=1}^{k-1} \mu_n v_A + \sum_{n=k}^{N} \mu_n \sqrt{\frac{x_A}{v_{Bn}}} v_A - x_A \tag{43}$$

with the right-hand derivative

$$\frac{\partial \Pi_A}{\partial x_A}^+ = \sum_{n=k}^{N} \mu_n \frac{v_A}{2\sqrt{x_A v_{Bn}}} - 1 \tag{44}$$

as x_A grows to v_{Bk}, the right-hand derivative decreases. Since

$$\lim_{x_A \xrightarrow{-} v_{Bk}} \frac{\partial \Pi_A}{\partial x_A}^+ = \sum_{n=k}^{N} \mu_n \frac{v_A}{2\sqrt{v_{Bk} v_{Bn}}} - 1 > \sum_{n=k+1}^{N} \mu_n \frac{v_A}{2\sqrt{v_{Bk} v_{Bn}}} - 1 = \lim_{x_A \xrightarrow{+} v_{Bk}} \frac{\partial \Pi_A}{\partial x_A}^+ \tag{45}$$

the right-hand derivative is a segmented monotone decreasing function. Since

$$\lim_{x_A \xrightarrow{+} 0} \frac{\partial \Pi_A}{\partial x_A}^+ = +\infty \tag{46}$$

$$\lim_{x_A \xrightarrow{-} \infty} \frac{\partial \Pi_A}{\partial x_A}^+ = -1 \tag{47}$$

Hence, Π_A always has a unique optimal solution. If $\frac{\partial \Pi_A}{\partial x_A}^+$ has intercept point x_A^* with X-axis, then x_A^* optimizes Π_A. Otherwise, there exists k such that

$$\lim_{x_A \xrightarrow{-} v_{Bk}} \frac{\partial \Pi_A}{\partial x_A}^+ > 0 > \lim_{x_A \xrightarrow{+} v_{Bk}} \frac{\partial \Pi_A}{\partial x_A}^+ \tag{48}$$

which means v_{Bk} optimizes Π_A. Hence, the expected total revenue is given by

$$\Pi(\mu) = \sum_{n=1}^{N} \mu_n \max(x_A, \sqrt{v_{Bn} x_A}) \tag{49}$$

□

Proof of Lemma 6.1. For the explicit form of interior solution, please refer to **Proposition 1** in ?. Since x_{BL}^* is the best response to x_A^*, that is, $x_{BL}^* > 0$ if and only if contestant A bids less than B's valuation, $x_A^* < v_L$:

$$\begin{aligned}
\Big(\frac{\frac{q}{\sqrt{v_L}} + \frac{1-q}{\sqrt{v_H}}}{\frac{1}{v_A} + \frac{q}{v_L} + \frac{1-q}{v_H}}\Big)^2 &< v_L \\
\frac{\frac{q}{\sqrt{v_L}} + \frac{1-q}{\sqrt{v_H}}}{\frac{1}{v_A} + \frac{q}{v_L} + \frac{1-q}{v_H}} &< \sqrt{v_L} \\
\frac{1-q}{\sqrt{v_H}} &< \sqrt{v_L}\Big(\frac{1}{v_A} + \frac{1-q}{v_H}\Big) \\
\frac{1-q}{\sqrt{v_H v_L}} &< \frac{1}{v_A} + \frac{1-q}{v_H} \\
(\frac{1}{\sqrt{v_H v_L}} - \frac{1}{v_H})q &> \frac{1}{\sqrt{v_H v_L}} - \frac{1}{v_A} - \frac{1}{v_H} \\
q &> 1 - \frac{\frac{1}{v_A}}{\frac{1}{\sqrt{v_H v_L}} - \frac{1}{v_H}} \\
q &> 1 - \frac{v_H\sqrt{v_L}}{v_A(\sqrt{v_H} - \sqrt{v_L})}
\end{aligned} \tag{50}$$

□

Proof. When $0 < q < 1 - \frac{v_H\sqrt{v_L}}{v_A(\sqrt{v_H}-\sqrt{v_L})}$, the optimal strategy for the low type contestant B will be $x_{BL}^* = 0$ according to Lemma 6.1. Hence, for contestant A, he maximizes

$$\Pi_A = qv_A + \frac{(1-q)x_A}{x_A + x_{BH}} v_A - x_A \tag{51}$$

The first order condition gives us

$$\frac{(1-q)x_{BH}v_A}{(x_A + x_{BH})^2} = 1 \tag{52}$$

and the high type contestant B maximizes

$$\Pi_{BH} = \frac{x_A}{x_A + x_{BH}} v_B - x_{BH} \tag{53}$$

The first order condition gives us

$$\frac{x_A v_H}{(x_A + x_{BH})^2} = 1 \Longrightarrow x_{BH} = \sqrt{x_A v_H} - x_A \tag{54}$$

Combining these two equations together, we have

$$\begin{aligned}(1-q)(\sqrt{x_A v_H} - x_A)v_A &= x_A v_H \\ (1-q)\sqrt{x_A v_H} v_A &= (1-q)x_A v_A + x_A v_H \\ \frac{(1-q)\sqrt{v_H} v_A}{(1-q)v_A + v_H} &= \sqrt{x_A} \\ x_A^* &= \Big(\frac{\frac{1-q}{\sqrt{v_H}}}{\frac{1}{v_A} + \frac{1-q}{v_H}}\Big)^2\end{aligned} \tag{55}$$

When $0 < q < 1 - \frac{v_H \sqrt{v_L}}{v_A(\sqrt{v_H} - \sqrt{v_L})}$,

$$\frac{\frac{q}{\sqrt{v_L}} + \frac{1-q}{\sqrt{v_H}}}{\frac{1}{v_A} + \frac{q}{v_L} + \frac{1-q}{v_H}} > \sqrt{v_L}$$

and $\frac{\frac{q}{\sqrt{v_L}}}{\frac{q}{v_L}} = \sqrt{v_L}$, we have

$$\frac{\frac{1-q}{\sqrt{v_H}}}{\frac{1}{v_A} + \frac{1-q}{v_H}} > \frac{\frac{q}{\sqrt{v_L}} + \frac{1-q}{\sqrt{v_H}}}{\frac{1}{v_A} + \frac{q}{v_L} + \frac{1-q}{v_H}} > \sqrt{v_L} \tag{56}$$

which means $x_A^* > v_L$, $x_{BL}^* = 0$. This forms a Bayesian Nash equilibrium. □

Proof of Theorem 6.3. By previous research done by Zhang and Zhou (2016)Zhang and Zhou (2016), Bayesian persuasion cannot yield higher effort when the solution is interior. When $1 - \frac{v_H \sqrt{v_L}}{v_A(\sqrt{v_H} - \sqrt{v_L})} > 0$, i.e.,

$$v_A > \frac{v_H \sqrt{v_L}}{\sqrt{v_H} - \sqrt{v_L}} > \sqrt{v_H v_L} \tag{57}$$

for some values of q, there exist boundary solutions such that $x_{BL}^* = 0$. When $v_A > \frac{v_H \sqrt{v_L}}{\sqrt{v_H} - \sqrt{v_L}}$ and $q < q^* = 1 - \frac{v_H \sqrt{v_L}}{v_A(\sqrt{v_H} - \sqrt{v_L})}$, according to Lemma 6.2 the equilibrium strategy for contestant A will be

$$x_A^* = \Big(\frac{\frac{1-q}{\sqrt{v_H}}}{\frac{1}{v_A} + \frac{1-q}{v_H}}\Big)^2 > v_L \tag{58}$$

When $q < q^*$, the expected total effort can be expressed as

$$\begin{aligned}\phi(q) &= qx_A^* + (1-q)\sqrt{x_A^* v_H} \\ &= q\Big(\frac{\frac{1-q}{\sqrt{v_H}}}{\frac{1}{v_A}+\frac{1-q}{v_H}}\Big)^2 + (1-q)^2\Big(\frac{1}{\frac{1}{v_A}+\frac{1-q}{v_H}}\Big) \\ &= \frac{q(1-q)^2 v_A v_H^2}{(v_H+v_A-qv_A)^2} + \frac{(1-q)^2 v_A v_H}{v_H+v_A-qv_A} \\ &= \frac{(1-q)^2 v_A v_H(qv_H+v_H+v_A-qv_A)}{(v_H+v_A-qv_A)^2}\end{aligned} \tag{59}$$

We re-scale the magnitude of v_A, v_H, v_L such that v_L=1. After the re-scaling, let A, H denote the relative magnitude of v_A and v_H, respectively. We can write $\phi(q)$ as

$$\phi(q) = \frac{(1-q)^2 AH((1+q)H + (1-q)A)}{(H+A-Aq)^2} \tag{60}$$

Any point located at the line segment between $(0, \phi(0))$ and $(1, \phi(1))$ in Cartesian coordinate system with x-coordinate q has the y-coordinate $\phi(0) + q(\phi(1) - \phi(0))$. Easily we get $\phi(0) = \frac{AH}{A+H}$ and $\phi(1) = \frac{A}{A+1}$

$$\begin{aligned}\phi(0) + q(\phi(1) - \phi(0)) &= \phi(1) - (1-q)(\phi(1) - \phi(0)) \\ &= \frac{A}{A+1} - (1-q)\frac{A^2(1-H)}{(A+H)(A+1)} \\ &= \frac{A^2 + AH - (1-q)A^2(1-H)}{(A+H)(A+1)} \\ &= \frac{qA^2 + AH + (1-q)A^2 H}{(A+H)(A+1)}\end{aligned} \tag{61}$$

We need to prove that for $q \le q^*$,

$$\frac{qA^2 + AH + (1-q)A^2 H}{(A+H)(A+1)} > \frac{(1-q)^2 AH((1+q)H + (1-q)A)}{(H+A-Aq)^2} \tag{62}$$

after a series of re-arrangement,

$$
\begin{aligned}
&\frac{qA^2 + AH + (1-q)A^2H}{(A+H)(A+1)} > \frac{(1-q)^2AH((1+q)H + (1-q)A)}{(H+A-Aq)^2} \\
\iff\ & (qA^2 + AH + (1-q)A^2H)(H + (1-q)A)^2 \\
&>(1-q)^2AH((1+q)H + (1-q)A)(A+H)(A+1) \\
\iff\ & (qA^2 + AH + (1-q)A^2H)(H^2 + 2(1-q)AH + (1-q)^2A^2) \\
&>(1-q)^2((1+q)H + (1-q)A)(A^3H + A^2H^2 + A^2H + AH^2) \\
\iff\ & (1-q)^3A^4H + q(1-q)^2A^4 + 2(1-q)^2A^3H^2 + (1-q^2)A^3H + (1-q)A^2H^3 + (2-q)A^2H^2 + AH^3 \\
&>(1-q)^3(A^4H + A^3H^2 + A^3H + A^2H^2) + (1-q)^2(1+q)(A^3H^2 + A^2H^3 + A^2H^2 + AH^3) \\
\iff\ & q(1-q)^2A^4 + (1-q^2)A^3H + q^2(1-q)A^2H^3 + (2-q)A^2H^2 + AH^3 \\
&>(1-q)^3(A^3H + A^2H^2) + (1-q)^2(1+q)(A^2H^2 + AH^3) \\
\iff\ & q(1-q)^2A^4 + (1-q^2)A^3H + q^2(1-q)A^2H^3 + (2-q)A^2H^2 + AH^3 \\
&>2(q^2 - 2q + 1)A^2H^2 + (-q^3 + 3q^2 - 3q + 1)A^3H + (q^3 - q^2 - q + 1)AH^3 \\
\iff\ & q(1-q)^2A^4 + (q^3 - 4q^2 + 3q)A^3H + q^2(1-q)A^2H^3 - (2q^2 - 3q)A^2H^2 - (q^3 - q^2 - q)AH^3 > 0 \\
\iff\ & q(1-q)^2A^4 + q(3-q)(1-q)A^3H + q^2(1-q)A^2H^3 + q(3-2q)A^2H^2 + (q + q^2 - q^3)AH^3 > 0
\end{aligned}
\tag{63}
$$

which is positive trivially. We also have for any $q \in [0, q^*]$

$$\phi(0) + q^*(\phi(1) - \phi(0)) > \phi(q^*) \tag{64}$$

where $q^* = 1 - \frac{v_H\sqrt{v_L}}{v_A(\sqrt{v_H} - \sqrt{v_L})}$. Then, since $\phi(q^*) < \phi(0) + q^*(\phi(1) - \phi(0))$, for $q \in [q^*, 1]$,

$$\phi(q) < (1-q)\phi(0) + q\phi(1) \tag{65}$$

Therefore, for any $q \in [0, 1] = [0, q^*] \cup [q^*, 1]$, **full disclosure condition** holds. □

Proof of Theorem 5.5. Let x_1^* denote the solution of the first order condition in Lemma 5.3 and x_2^* denote the solution of the first order condition in Lemma 5.4. We have

$$x_1^* = \frac{v_A^2}{4}\Big(\frac{q}{\sqrt{v_L}} + \frac{1-q}{\sqrt{v_H}}\Big)^2 \tag{66}$$

and

$$x_2^* = \frac{v_A^2}{4}\Big(\frac{1-q}{\sqrt{v_H}}\Big)^2 \tag{67}$$

It is obvious that

$$x_2^* < x_1^* \tag{68}$$

Combining the observation that in both segments, Π_A is concave, the optimal strategy of contestant

A is

$$x_A = \begin{cases} x_1^* & \textbf{if } x_1^* < v_L \\ v_L & \textbf{if } x_2^* \leq v_L \leq x_1^* \\ x_2^* & \textbf{if } v_L < x_2^* < v_H \\ v_H & \textbf{if } x_2^* \geq v_H \end{cases} \tag{69}$$

let q_1 denote the threshold value when $x_1^* = v_L$, q_2 denote the threshold value when $x_2^* = v_L$ and q_3 denote the threshold value when $x_2^* = v_H$. Results of q_2 and q_3 are straightforward,

$$q_2 = 1 - \frac{2\sqrt{v_H v_L}}{v_A}, q_3 = 1 - \frac{2v_H}{v_A}$$

Here we only derive q_1,

$$\begin{aligned} \frac{v_A^2}{4}\Big(\frac{q_1}{\sqrt{v_L}} + \frac{1-q_1}{\sqrt{v_H}}\Big)^2 &= v_L \\ \frac{v_A}{2}\Big(\frac{q_1}{\sqrt{v_L}} + \frac{1-q_1}{\sqrt{v_H}}\Big) &= \sqrt{v_L} \\ \frac{q_1}{\sqrt{v_L}} + \frac{1-q_1}{\sqrt{v_H}} &= \frac{2\sqrt{v_L}}{v_A} \\ \frac{q_1}{\sqrt{v_L}} - \frac{q_1}{\sqrt{v_H}} &= \frac{2\sqrt{v_L}}{v_A} - \frac{1}{\sqrt{v_H}} \end{aligned}$$

The result is

$$q_1 = \frac{\sqrt{v_L}(2\sqrt{v_H v_L} - v_A)}{(\sqrt{v_H} - \sqrt{v_L})v_A}$$

At last, since q_1, q_2, q_3 may exceed interval $[0, 1]$, we need to calculate the value range of v_A given v_H and v_L. The result is summarized in the following table. Combine the value of q_1, q_2, q_3, this

q	≥ 0	≤ 1
q_1	$v_A \leq 2\sqrt{v_H v_L}$	$v_A \geq 2v_L$
q_2	$v_A \geq 2\sqrt{v_H v_L}$	always
q_2	$v_A \geq 2v_H$	always

Table 1: Existence of q

table as well as Equation 69 we can conclude the result. □

Proof of Proposition 5.8. If $q_1 \leq \frac{1}{2}$, the derivatives are positive when $q < q_1$ and negative when $q > q_1$. Since $\phi(q)$ is continuous at $q = q_1$, the concave closure of $\phi(q)$ is itself. Re-write condition $q_1 = \frac{\sqrt{v_L}(2\sqrt{v_H v_L} - v_A)}{(\sqrt{v_H} - \sqrt{v_L})v_A}) \leq \frac{1}{2}$ as

$$v_A \geq \frac{4\sqrt{v_H}v_L}{\sqrt{v_H} + \sqrt{v_L}} \tag{70}$$

since $\frac{4\sqrt{v_H}v_L}{\sqrt{v_H}+\sqrt{v_L}}$ is obviously located inside $(2v_L, 2\sqrt{v_H v_L})$. No disclosure is optimal among all possible signals when $v_A \in [\frac{4\sqrt{v_H}v_L}{\sqrt{v_H}+\sqrt{v_L}}, 2\sqrt{v_H v_L}]$.

If $v_A \in [2v_L, \frac{4\sqrt{v_H}v_L}{\sqrt{v_H}+\sqrt{v_L}})$, we need to compare the left-hand derivative and the right-hand derivative, which is $v_L - \sqrt{v_H v_L}$, at inflection point $q = q_1$. If

$$\frac{\partial\phi^-}{\partial q} - \frac{\partial\phi^+}{\partial q} > 0 \tag{71}$$

we can conclude that the concave disclosure of $\phi(q)$ is itself. The left-hand derivative is computed as

$$\begin{aligned}
\frac{\partial\phi^-}{\partial q} &= \frac{v_A}{2}\left(1 - \frac{2\sqrt{v_L}(2\sqrt{v_H v_L} - v_A)}{(\sqrt{v_H} - \sqrt{v_L})v_A}\right)\frac{v_H + v_L - 2\sqrt{v_H v_L}}{\sqrt{v_H v_L}} \\
&= \frac{(\sqrt{v_H} - \sqrt{v_L})v_A - 2\sqrt{v_L}(2\sqrt{v_H v_L} - v_A)}{2(\sqrt{v_H} - \sqrt{v_L})}\frac{(\sqrt{v_H} - \sqrt{v_L})^2}{\sqrt{v_H v_L}} \\
&= \frac{(\sqrt{v_H} + \sqrt{v_L})v_A - 4\sqrt{v_H}v_L}{2}\frac{\sqrt{v_H} - \sqrt{v_L}}{\sqrt{v_H v_L}} \\
&= \frac{(v_H - v_L)v_A - 4v_H v_L + 4v_H^{0.5}v_L^{1.5}}{2\sqrt{v_H v_L}}
\end{aligned} \tag{72}$$

Finally,

$$\begin{aligned}
\frac{\partial\phi^-}{\partial q} - \frac{\partial\phi^+}{\partial q} &= \frac{(v_H - v_L)v_A - 4v_H v_L + 4v_H^{0.5}v_L^{1.5}}{2\sqrt{v_H v_L}} - \left(v_L - \sqrt{v_H v_L}\right) \\
&= \frac{(v_H - v_L)v_A - 2v_H v_L + 2v_H^{0.5}v_L^{1.5}}{2\sqrt{v_H v_L}} \\
&= \frac{(v_H - v_L)v_A - 2\sqrt{v_H v_L}(\sqrt{v_H v_L} - v_L)}{2\sqrt{v_H v_L}} \\
&\geq \frac{(v_H - v_L)2v_L - 2\sqrt{v_H v_L}(\sqrt{v_H v_L} - v_L)}{2\sqrt{v_H v_L}} \\
&= \frac{-2v_L^2 + 2\sqrt{v_H v_L}v_L}{2\sqrt{v_H v_L}} \\
&= \frac{2v_L(\sqrt{v_H v_L} - v_L)}{2\sqrt{v_H v_L}} \\
&> 0
\end{aligned} \tag{73}$$

□

Proof of Proposition 5.9. When $q > q_2$, contestant A will exert v_L, hence the expected total effort is $\phi_2(q)$ with slope $v_L - \sqrt{v_H v_L}$. Since $\phi_2'(q) = v_L - \sqrt{v_H v_L} > \phi(1) - \phi(0)$, **full disclosure condition** holds when $q \in [q_2, 1]$.

Let us focus on the left-side of $\phi(q)$. Since $\phi'''(q) = \frac{3v_A^2}{2v_H} > 0$, we have $\lim_{q \xrightarrow{+} 0} \phi''(q) = \phi''(0) < \phi''(q_2) = \lim_{q \xrightarrow{-} q_2} \phi''(q)$. There are three different cases,

(1) $0 \leq \phi_3''(0) < \phi_3''(q_2)$, $\phi_3(q)$ is convex here. **Full disclosure condition** holds.

(2) $\phi_3''(0) < \phi_3''(q_2) < 0$, $\phi_3(q)$ is concave here. Simple algebra gives us $\phi_3'(q) < \phi_3'(0) < \phi(1) - \phi(0)$. Integrating both sides between $[0, q]$ we have

$$\phi_3(q) - \phi_3(0) < q(\phi(1) - \phi(0)) \tag{74}$$

Full disclosure condition holds.

(3) $\phi_3''(0) < 0 < \phi_3''(q_2)$, let q^* denote the solution to $\phi_3''(q) = 0$. According to the analysis of **(2)**, we have $\phi_3(q) - \phi_3(0) < q(\phi(1) - \phi(0))$ when $q \leq q^*$. According to analysis of **(1)**, when $q \in (q^*, q_2]$, we have

$$\phi_3(q) < \phi_3(q^*) + (q - q^*)\frac{\phi_3(q_2) - \phi_3(q^*)}{q_2 - q^*} \tag{75}$$

Full disclosure condition holds.

Now, we specify the conditions for three different cases. The second order derivatives at two endpoints are computed as

$$\phi''(0) = \frac{v_A v_H - v_A^2}{v_H} \tag{76}$$

and

$$\phi''(q) = \frac{v_A}{2v_H}(2v_H + v_A - 6\sqrt{v_H v_L}) \tag{77}$$

Case (1) happens if and only if $v_A < v_H$, in other words, $v_H > 4v_L$ and $v_A \in (2\sqrt{v_H v_L}, v_H]$.

Case (2) happens if and only if $v_A < 6\sqrt{v_H v_L} - 2v_H$, in other words, $v_H \in (2.25v_L, 4v_L]$ and $v_A \in (2\sqrt{v_H v_L}, 6\sqrt{v_H v_L} - 2v_H]$ **or** $v_H \leq 2.25v_L$ and $v_A \in (2\sqrt{v_H v_L}, 2v_H]$. And $\phi(q)$ is concave when $q \in [0, \frac{2v_A - 2v_H}{3v_A}]$ and convex when $q \in [\frac{2v_A - 2v_H}{3v_A}, q_2]$.

Case (3) happens if and only if $v_H > 4v_L$ and $v_A \in [v_H, 2v_H]$ **or** $v_H \in (2.25v_L, 4v_L]$ and $v_A \in (6\sqrt{v_H v_L} - 2v_H, 2v_H]$.

From the above analysis, we find out the dominance of full disclosure when $v_A \in (2\sqrt{v_H v_L}, 2v_H]$. $\square$

Proof of Theorem 5.10. Similar to the proof of Proposition 5.9, $\phi(q_2)$ is less than $\phi(0) + q_2(\phi(1) - \phi(0))$. By Equation 25, we have

$$\lim_{q \xrightarrow{+} 1 - \frac{2v_H}{v_A}} \phi'(q) = v_H - v_A < 0 \tag{78}$$

and

$$\lim_{q \xrightarrow{-} 1 - \frac{2\sqrt{v_H v_L}}{v_A}} \phi'(q) = -2\sqrt{v_H v_L} - \frac{\sqrt{v_L} v_A}{\sqrt{v_H}} + 3v_L < v_L - \sqrt{v_H v_L} \tag{79}$$

As for the second order derivative, we have the following observations

$$\lim_{q \xrightarrow{+} 1-\frac{2v_H}{v_A}} \phi''(q) = \frac{v_A^2 - 4v_A v_H}{2v_H} \tag{80}$$

$$\lim_{q \xrightarrow{-} 1-\frac{2\sqrt{v_H v_L}}{v_A}} \phi''(q) = \frac{v_A}{2v_H}(2v_H + v_A - 6\sqrt{v_H v_L}) \tag{81}$$

We can classify them into three different cases according to the signs of $\lim_{q \xrightarrow{+} 1-\frac{2v_H}{v_A}} \phi''(q)$ and $\lim_{q \xrightarrow{-} 1-\frac{2\sqrt{v_H v_L}}{v_A}} \phi''(q)$, as it is shown in the following proposition. Firstly, let us look at the concavity and convexity when $v_H \geq 2.25v_L$. If $v_H \geq 2.25v_L$ and $v_A \in (2v_H, 4v_H]$, total effort $\phi(q)$ is concave among $(1 - \frac{2v_H}{v_A}, \frac{2v_A - 2v_H}{3v_A})$ and convex among $(\frac{2v_A-2v_H}{3v_A}, 1 - \frac{2\sqrt{v_H v_L}}{v_A})$. If $v_H \geq 2.25v_L$ and $v_A \in (4v_H, +\infty)$, total effort $\phi(q)$ is convex among $(1 - \frac{2v_H}{v_A}, 1 - \frac{2\sqrt{v_H v_L}}{v_A})$.

Secondly, let us look at the concavity and convexity when $v_H < 2.25v_L$. If $v_H < 2.25v_L$ and $v_A \in [2v_H, 6\sqrt{v_H v_L} - 2v_H]$, total effort $\phi(q)$ is concave among $(1 - \frac{2v_H}{v_A}, 1 - \frac{2\sqrt{v_H v_L}}{v_A})$. If $v_H < 2.25v_L$ and $v_A \in (6\sqrt{v_H v_L} - 2v_H, 4v_H]$, total effort $\phi(q)$ is concave among $(1 - \frac{2v_H}{v_A}, \frac{2v_A - 2v_H}{3v_A})$ and convex among $(\frac{2v_A-2v_H}{3v_A}, 1 - \frac{2\sqrt{v_H v_L}}{v_A})$. If $v_H < 2.25v_L$ and $v_A \in (4v_H, +\infty)$, total effort $\phi(q)$ is convex among $(1 - \frac{2v_H}{v_A}, 1 - \frac{2\sqrt{v_H v_L}}{v_A})$.

One interesting fact is that

$$\begin{aligned} & \phi'(q_3) < \frac{\phi(1) - \phi(q_3)}{1 - q_3} \\ \iff \quad & v_H - v_A < \frac{v_L - v_H}{\frac{2v_H}{v_A}} \\ \iff \quad & 2v_H(v_H - v_A) < v_A(v_L - v_H) \\ \iff \quad & v_H(2v_H - v_A) < 0 < v_A v_L \end{aligned} \tag{82}$$

Equation 82 simplifies the analysis to a great extent, that is, the concave closure of $\phi(q)$, named as $\mathbf{cav}\phi(q)$ is a piecewise linear function,

$$\mathbf{cav}\phi(q) = \begin{cases} v_H & q \leq q_3 \\ v_L + (1-q)\frac{v_A(v_H - v_L)}{2v_H} & q > q_3 \end{cases} \tag{83}$$

□

Proof of Theorem 7.1. **(1)** $v_A < 2v_L$

$$
\begin{aligned}
\Pi_B^N(q) &= q\Pi_B(\frac{v_A^2}{4}(\frac{q}{\sqrt{v_L}} + \frac{1-q}{\sqrt{v_H}})^2, v_L) + (1-q)\Pi_B(\frac{v_A^2}{4}(\frac{q}{\sqrt{v_L}} + \frac{1-q}{\sqrt{v_H}})^2, v_H) \\
&= q\Big(\frac{v_A}{2}(\frac{q}{\sqrt{v_L}} + \frac{1-q}{\sqrt{v_H}}) - \sqrt{v_L}\Big)^2 + (1-q)\Big(\frac{v_A}{2}(\frac{q}{\sqrt{v_L}} + \frac{1-q}{\sqrt{v_H}}) - \sqrt{v_H}\Big)^2 \\
&= \frac{v_A^2}{4}(\frac{q}{\sqrt{v_L}} + \frac{1-q}{\sqrt{v_H}})^2 + qv_L + (1-q)v_H - v_A(\frac{q}{\sqrt{v_L}} + \frac{1-q}{\sqrt{v_H}})(q\sqrt{v_L} + (1-q)\sqrt{v_H})
\end{aligned}
$$

Analysis of (1)

The second order condition gives us

$$
\frac{\partial^2 \Pi_B^N(q)}{\partial q^2} = \frac{v_A^2}{2}(\frac{1}{\sqrt{v_L}} - \frac{1}{\sqrt{v_H}})^2 + 2v_A(\frac{1}{\sqrt{v_L}} - \frac{1}{\sqrt{v_H}})(\sqrt{v_H} - \sqrt{v_L}) > 0
$$

Since $\Pi_B^N(0) = \Pi_B^S(0)$ and $\Pi_B^N(1) = \Pi_B^S(1)$, therefore, contestant B has incentive to fully share information.

(2) $v_A \in [2v_L, 2\sqrt{v_H v_L})$

(2-1) $q \in [0, q_1)$

The expected utility function has the same expression as $v_A < 2v_L$ case and hence a convex function inside $q \in [0, q_1)$.

(2-2) $q \in [q_1, 1]$

The expected utility function is a linear function,

$$
\Pi_B^N(q) = q\Pi_B(v_L, v_L) + (1-q)\Pi_B(v_L, v_H) = (1-q)(\sqrt{v_H} - \sqrt{v_L})^2
$$

Analysis of (2)

The slope is $-(\sqrt{v_H} - \sqrt{v_L})^2$ inside $q \in [q_1, 1]$. While the slope of $\Pi_B^S(q)$ is $-v_H(1 - \frac{v_A}{2v_H})^2$,

$$
\begin{aligned}
\frac{\partial \Pi_B^N(q)}{\partial q} - \frac{\partial \Pi_B^S(q)}{\partial q} &= -(\sqrt{v_H} - \sqrt{v_L})^2 + v_H(1 - \frac{v_A}{2v_H})^2 \\
&= 2\sqrt{v_H v_L} - v_H - v_L + v_H - v_A + \frac{v_A^2}{4v_H} \\
&= (2\sqrt{v_H v_L} - v_A) - \frac{4v_H v_L - v_A^2}{4v_H} \\
&= \frac{(2\sqrt{v_H v_L} - v_A)(4v_H - 2\sqrt{v_H v_L} - v_A)}{4v_H} > 0
\end{aligned}
$$

Contestant B has incentive to share information because $\Pi_B^N(q) \le \Pi_B^S(q)$ for all q.

(3) $v_A \in [2\sqrt{v_H v_L}, 2v_H)$

(3-1) $q \in [0, q_2)$

$$\begin{aligned}\Pi_B^N(q) &= (1-q)\Pi_B(\frac{v_A^2}{4}(\frac{1-q}{\sqrt{v_H}})^2, v_H) \\ &= (1-q)(\frac{v_A}{2}(\frac{1-q}{\sqrt{v_H}}) - \sqrt{v_H})^2 \\ &= \frac{v_A^2(1-q)^3}{4v_H} - v_A(1-q)^2 + (1-q)v_H\end{aligned}$$

(3-2) $q \in [q_2, 1]$

$$\Pi_B^N(q) = q\Pi_B(v_L, v_L) + (1-q)\Pi_B(v_L, v_H) = (1-q)(\sqrt{v_H} - \sqrt{v_L})^2$$

Analysis of (3)

$\Pi_B^N(q)$ has two parts. When $q \in [q_2, 1]$, it is a linear function. The slope is $-(\sqrt{v_H} - \sqrt{v_L})^2$ inside $q \in [q_2, 1]$. The slope of $\Pi_B^S(q)$ is $-v_H(1 - \frac{v_A}{2v_H})^2$,

$$\begin{aligned}\frac{\partial \Pi_B^N(q)}{\partial q} - \frac{\partial \Pi_B^S(q)}{\partial q} &= -(\sqrt{v_H} - \sqrt{v_L})^2 + v_H(1 - \frac{v_A}{2v_H})^2 \\ &= \frac{(2\sqrt{v_H v_L} - v_A)(4v_H - 2\sqrt{v_H v_L} - v_A)}{4v_H} < 0\end{aligned}$$

Therefore, $\Pi_B^N(q) > \Pi_B^S(q)$ for $q \in [q_2, 1]$.

When $q \in [0, q_2)$,

$$\begin{aligned}\frac{\partial \Pi_B^N(q)}{\partial q} - \frac{\partial \Pi_B^S(q)}{\partial q} &= -\frac{3v_A^2(1-q)^2}{4v_H} + 2v_A(1-q) - v_H + v_H(1 - \frac{v_A}{2v_H})^2 \\ &= -\frac{v_A(1-q)(3v_A(1-q) - 8v_H)}{4v_H} - v_A + \frac{v_A^2}{4v_H} < 0\end{aligned}$$

indicating that $\Pi_B^N(q) > \Pi_B^S(q)$ for $q \in [0, q_2)$.

Therefore, contestant B has no incentive to share information.

(4) $v_A \geq 2v_H$

(4-1) $q \in [0, q_3]$

$$\Pi_B^N(q) = q\Pi_B(v_H, v_L) + (1-q)\Pi_B(v_H, v_H) = 0$$

(4-2) $q \in (q_3, q_2)$

$$\Pi_B^N(q) = (1-q)\Pi_B(\frac{v_A^2}{4}(\frac{1-q}{\sqrt{v_H}})^2, v_H) = (1-q)(\frac{v_A}{2}(\frac{1-q}{\sqrt{v_H}}) - \sqrt{v_H})^2$$

(4-3) $q \in (q_2, 1]$

$$\Pi_B^N(q) = q\Pi_B(v_L, v_L) + (1-q)\Pi_B(v_L, v_H) = (1-q)(\sqrt{v_H} - \sqrt{v_L})^2$$

Analysis of (4)

Contestant B has no incentive to share information because $\Pi_B^N(q) \geq 0 = \Pi_B^S(q)$. $\square$

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