A Fairness Condition for Unfair Contests: Multi-Dimensional Favoritism with Asymmetric Players*

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This Version: May 9, 2016
Initial Version: November 29, 2014

Abstract

We study optimal contest design when two asymmetric players compete in a complete-information all-pay-auction contest with multi-dimensional favoritism. The favoritism can take the form of a player being given a head-start advantage ("additive bias"), or an effectiveness advantage which enhances each unit of effort ("multiplicative bias") expended by a player. We solve for the optimal contest for a designer who would like to achieve each of the following objectives: maximizing the total effort, maximizing the total score, maximizing the highest effort by a single player, and maximizing the winner's effort. We find a single necessary "fairness condition" under each of the four objectives of the designer, and further show that the condition is robust to non-linear forms of favoritism. Our

*This research is supported by National Science Foundation of China Young Scholar Research Grant (grant number: 71203112) and Tsinghua University Independent Research Project Grant (grant number: 2012Z02181). For helpful comments, we thank Qian Jiao, Hao Li, Jingfeng Lv, Kang Rong, Qianfeng Tang, Zhewei Wang, Wenzhang Zhang, Yongchao Zhang, Jun Yu and Junjie Zhou, as well as participants in the Shanghai Economic Theory Workshop (2014). The current version of this paper is very preliminary. Please do not quote without the permission from the authors.

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findings contribute to a better understanding of the optimal unfair contests, and generalize some of the results in the previous literature with single-dimensioned unfairness.

JEL Classification Codes: C72, D44

Keywords: All-Pay Auctions, Contests, Multi-Dimensional Discrimination, Favoritism, Fairness, Handicap, Head Start

1 Introduction

Motivating individual effort under differences in abilities is a familiar challenge with widespread applications. For example, a parent wants to encourage each of their children to perform at their best in schoolwork, despite differences in scholastic abilities and inclinations. An advisor has students who differ in their motivation towards research, but would like to encourage each of them to produce the best possible thesis. A manager is aware that his employees differ in their skill levels, but would nevertheless like to induce the best efforts from each of them in order to maximize company profits. When such scenarios involve competition in the form of awarding a prize for the best-performing individual, it may be optimal for the parent, the advisor, and the manager, to implement some forms of favorism in order to maintain the motivations of the individuals they are supervising.

Contest theory provides a canonical game theoretic model of strategic interaction among individuals for a prize. How can a contest designer allocate favors among participants with heterogeneous abilities to achieve his or her objectives? We study optimal contest design when two asymmetric players compete in a complete-information all-pay-auction contest with the possibility of favoritism. In our benchmark framework, the favoritism can take the form of a player being given a head-start advantage ("additive bias"), or an effectiveness advantage which enhances each unit of effort ("multiplicative bias") expended by a player. We derive the optimal contest for a contest designer who would like to achieve each of the following plausible objectives: maximizing the total effort, maximizing the total score, maximizing the highest effort by a single player, and maximizing the winner's effort. A "fairness condition" is derived, which is necessary for the contest designer in setting the optimal contest under all four objectives, and which is also show to be robust to non-linear forms of favoritism. Our findings contribute to a more complete understanding of the optimal unfair contests, and also generalize some of the results in the previous literature which has largely focused on unfairness of a specific type.

Our paper contributes to the literature on unfair contests, where the most closely relevant study to ours is Kirkegaard (2012). Kirkegaard (2012) models the contest as an all-pay auction with incomplete information, in which two bidders, one "strong" and one "weak", are independently and privately informed about their valuation of the prize. Bidders are risk-neutral and the cost of bidding is linear in the bid. Whereas

Kirekgaard (2012) focuses on the case of incomplete information where the contest designer would like to maximize total effort, we focus on the complete information setting, considering four objectives of the contest designer: maximizing total effort, maximizing total scores, maximizing the highest effort, and maximizing the winning effort. Futhermore, we consider more general non-linear forms of fairness, and show that our fairness condition is still a necessary condition under these more generalized forms of unfairness, in addition to the case of endogenized potential additive and multiplicative biases. Other closely related studies are Li and Yu (2012), Fu (2006) and Konrad (2002). Li and Yu (2012) consider additive bias and multiplicative bias each separately under a complete information contest and show that additive bias is strictly preferred by the contest designer for efficiency purpises. Our model generalizes their findings by allowing the contest designer to pursue more general forms of favoritism, including both additive and multiplicative, as well as non-linear forms of favoritism, and shows the conditions under which multiplicative bias can be useful. In formulating a model of affirmative action in college admissions, Fu (2006) endogenizes multiplicative bias in the model with two heterogeneous bidders, showing that it is optimal to handicap the strong bidder. Our model generalizes Fu (2006) by endogenizing both multiplicative and additive bias, and providing a general condition for "fairness" under unequal ability endowments. Konrad (2002) shows the equilibrium strategy of a two-bidder model with additive and multiplicative bias which are exogenous in the model, when the bidders are symmetric. Our model generalizes these findings to the case of asymmetric bidders, while also endogenizing the form of bias implemented by the contest designer.

Our results provide new insights about how to ideally set up an unfair contest, which can serve to equalize the incentives for effort when participants differ in their abilities in the competition. We show that in order to incentivize both contestants under all four potential objectives, the designer will apply the multiplicative favoritism bias to the more capable contestant, while applying the additive favoritism bias to the less capable contestant. That is, to optimally allocate favoritism, the less capable contestant should be given a lump-sum head start in scores, while the more capable contestant should receive score subsidies per unit of effort expended. We derive a generalized necessary fairness condition for the designer which is new to the literature. This condition states that the designer should set the favors such that when both players make effort levels equal to their valuations of the prize, they receive the same score or outcome in the contest.

To summarize our contribution relative to the existing literature, we solve for the optimal contest where both additive bias and multiplicative bias are adjustable by the contest designer. Our findings generalize the results in Li and Yu (2012), which studies the optimal contest with either additive bias or multiplicative bias separately. We fully analyze the complete-information asymmetric all-pay auctions with headstarts and handicaps, while Kirkegaard (2012) focuses on the incomplete-information case. We characterize the equilibrium effort levels for asymmetric all-pay auctions, which generalizes the result in Konrad (2002) for symmetric bidders. In addition, we consider the optimal contest design for four different but reasonable objectives of the contest designer, including maximizing total effort, maximizing total score, maximizing the highest effort, and maximizing the winning effort. We also consider a general setup where the favoritism does not necessarily take the linear form implied by additive and multiplicative biases, showing that our fairness condition still holds.

The remainder of the paper is organized as follows. Section 2 describes the model setup and characterizes the equilibrium strategy of the players. In Section 3 we solve for the optimal contest with endogenized additive bias and multiplicative bias when the contest designer would like to maximize the total effort. In Section 4 the potential alternative objectives of the designer including maximizing total score, maximizing the highest effort, and maximizing the winning effort are analyzed, and the optimal contest under each objective is derived. Section 5 provides a general setup where the favoritism does not necessarily take the forms of either headstarts or handicaps, but could be non-linear in form, and shows that the fairness condition still holds. Section 6 concludes and discusses.

2 Model and Equilibrium

2.1 The Model Setup

Two players (labeled 1 and 2) participate in an all-pay-auction contest. Player i's valuation of the prize is v_i, where $v_i \in (0, +\infty)$, $i = 1, 2$. We assume $v_1 \geq v_2$ such that player 1 tends to value the contest more highly, which represents being the more capable or naturally motivated player. Player i's effort level is denoted by x_i, with $x_i \in [0, +\infty)$, $i = 1, 2$.

The contest designer is able to apply favoritism along two dimensions by potentially imposing an additive bias and/or multiplicative bias on the performance of the players. Player i's performance is measured by his score, denoted by s_i, with $s_i \in (-\infty, +\infty)$, $i = 1, 2$. Without loss of generality, assume $s_1 = \theta_1 v_1 + \theta_2$ and $s_2 = v_2$, where θ_1 refers to the multiplicative bias and θ_2 refers to the additive bias. By assuming $\theta_1 \in \left[\underline{\theta}_1, \bar{\theta}_1\right]$, where $0 < \underline{\theta}_1 < \frac{v_2}{v_1}$ and $\frac{v_1}{v_2} < \bar{\theta}_1 < +\infty$, and $\theta_2 \in (-\infty, +\infty)$, it is clear that: (1) when $\theta_1 > 1$, player 1 receives a *productivity advantage*; In other words, player 2 is *handicapped*, and (2) when $\theta_2 < 0$, player 2 receives a *headstart advantage*. Other allocations of favoritism can be described similarly.

In this all-pay-auction unfair contest, the player with the higher score (note, not necessarily the one with the higher effort) wins the prize, with equal splitting of the prize in the case of a tie, and no matter who wins the prize, both players pay a cost equal to their effort level. Thus, player i's payoff is $u_i(x_i) = P(s_i, s_j) v_i - x_i$, where

$$P(s_i, s_j) = \begin{cases} 1 & \text{if } s_i > s_j \\ \frac{1}{2} & \text{if } s_i = s_j \\ 0 & \text{if } s_i < s_j \end{cases}.$$

v_i and θ_i, as well as all rules of the game, are common knowledge among the two players, as in the standard complete information contest theory. Each player simultaneously chooses his own effort level x_i to maximize his expected payoff $u_i(x_i)$.

To avoid discussion of trivial cases where the contest is so unfairly set such that any given player can make zero effort and still win the contest, conditional on the other player at least making a profitable effort choice, we make the following two assumptions, derived via the winning condition for each player.

Assumption 1 (Participation Constraint for Player 1) *$\theta_1 v_1 + \theta_2 > 0$.*

Assumption 2 (Participation Constraint for Player 2) *$v_2 > \theta_2$.*

Assumption 1 implies that when player 2 makes no effort, player 1 can win the contest if he makes high enough effort. Similarly, Assumption 2 implies that when player 1 makes no effort, player 2 can win the contest if he makes high enough effort.

2.2 The Equilibrium Allocation

The equilibrium effort allocation for a symmetric unfair contest where $v_1 = v_2$ has already been derived in Konrad (2002). Here we characterize players' equilibrium strategies for the more general asymmetric unfair contest, which has been referred to, but not explicitly provided in the asymmetric information analysis of Kirkegaard (2012).

Theorem 1 *Under Assumptions 1 and 2, the all-pay-auction contest has the following mixed-strategy equilibrium $(F_1(x_1), F_2(x_2))$ for the following possible conditions on the valuation and favoritism parameters $(v_1, v_2, \theta_1, \theta_2)$:*

(i) $\theta_2 \leq 0$ *and* $\theta_1 v_1 + \theta_2 > v_2$*:* $F_1(x) = \begin{cases} 0 & \text{if } x \in [0, -\frac{\theta_2}{\theta_1}) \\ \frac{\theta_1 x + \theta_2}{v_2} & \text{if } x \in [-\frac{\theta_2}{\theta_1}, \frac{v_2 - \theta_2}{\theta_1}) \\ 1 & \text{if } x \in [\frac{v_2-\theta_2}{\theta_1}, +\infty) \end{cases}$, $F_2(x) = \begin{cases} 1 + \frac{x - v_2}{v_1 \theta_1} & \text{if } x \in [0, v_2) \\ 1 & \text{if } x \in [v_2, +\infty) \end{cases}$;

(ii) $\theta_2 \leq 0$ *and* $0 < \theta_1 v_1 + \theta_2 < v_2$*:* $F_1(x) = \begin{cases} 1 - \frac{\theta_1 v_1 + \theta_2}{v_2} & \text{if } x \in [0, -\frac{\theta_2}{\theta_1}) \\ 1 + \frac{\theta_1 x - \theta_1 v_1}{v_2} & \text{if } x \in [-\frac{\theta_2}{\theta_1}, v_1) \\ 1 & \text{if } x \in [v_1, +\infty) \end{cases}$,

$$F_2(x) = \begin{cases} \frac{x - \theta_2}{v_1 \theta_1} & x \in [0, \theta_1 v_1 + \theta_2) \\ 1 & x \in [\theta_1 v_1 + \theta_2, +\infty) \end{cases};$$

(iii) $0 < \theta_2 < v_2$ *and* $\theta_1 v_1 + \theta_2 > v_2$*:* $F_1(x) = \begin{cases} \frac{\theta_1 x + \theta_2}{v_2} & \text{if } x \in [0, \frac{v_2 - \theta_2}{\theta_1}) \\ 1 & \text{if } x \in [\frac{v_2 - \theta_2}{\theta_1}, +\infty) \end{cases}$,

$$F_2(x) = \begin{cases} 1 + \frac{\theta_2 - v_2}{v_1 \theta_1} & \text{if } x \in [0, \theta_2) \\ 1 + \frac{x - v_2}{v_1 \theta_1} & \text{if } x \in [\theta_2, v_2) \\ 1 & \text{if } x \in [v_2, +\infty) \end{cases};$$

(iv) $0 < \theta_2 < v_2$ *and* $0 < \theta_1 v_1 + \theta_2 < v_2$*:* $F_1(x) = \begin{cases} 1 + \frac{\theta_1 x - \theta_1 v_1}{v_2} & \text{if } x \in [0, v_1) \\ 1 & \text{if } x \in [v_1, +\infty) \end{cases}$,

$$F_2(x) = \begin{cases} 0 & \text{if } x \in [0, \theta_2) \\ \frac{x - \theta_2}{v_1 \theta_1} & \text{if } x \in [\theta_2, \theta_1 v_1 + \theta_2) \\ 1 & \text{if } x \in [\theta_1 v_1 + \theta_2, +\infty) \end{cases}.$$

Proof. See Appendix. ■

The equilibrium strategies can be derived by considering the relationship between the favoritism parameters and contestant valuations in four cases, corresponding to positive (iii and iv) versus negative (i and ii) additive bias, and strong (i and iii) versus weak (ii and iv) favoritism towards player 1. As in the symmetric case, in an asymmetric unfair contest with both additive bias and multiplicative bias, the equilibrium strategies for both players are in mixed strategies, as shown in Theorem 1.

Case (i) indicates that when player 2 receives a (weak) headstart advantage and player 1 has enough multiplicative favoritism to compensate, player 1 assigns zero probability weight to sufficiently high and sufficiently low effort levels, while having positive probability weight for an intermediate range of effort levels. Player 2 assigns positive probablity weight to profitable effort levels, while avoiding effort levels which are too high to be profitable. In case (ii),

Based on players' equilibrium strategies, it is straightforward to solve for the equilibrium payoff for each player. We summarize this result in the following proposition.

Proposition 1 (Equilibrium Payoffs) *Under Assumptions 1 and 2, players' equilibrium payoffs* (u_1, u_2) *in the all-pay-auction contest for different cases of valuation and favoritism parameters* $(v_1, v_2, \theta_1, \theta_2)$ *are the following:*

(1) If $\theta_1 v_1 + \theta_2 > v_2$ *then we have* $u_1 = \frac{\theta_1 v_1 + \theta_2 - v_2}{\theta_1} > 0$ *and* $u_2 = 0$*;*

(2) If $0 < \theta_1 v_1 + \theta_2 < v_2$ *then we have* $u_1 = 0$ *and* $u_2 = v_2 - \theta_1 v_1 - \theta_2 > 0$*.*

It is worth noting from the result in Proposition 1 that whether player 1 or player 2 has a higher equilibrium payoff, does not depend on who has a higher type alone: in case (2) $u_2 > u_1$ while $v_2 > v_1$. However, it does depend on which player receives favorable treatment in the sense of comparing $\theta_1 v_1 + \theta_2$ with v_2. This finding is given in the following corollary of Proposition 1.

Corollary 1 *The higher equilibrium payoff goes to the favored player, not necessarily the more capable one.*

The result in Corollary 1 can be captured by the phrase "better favored than capable", if we agree that the equation $\theta_1 v_1 + \theta_2 = v_2$ represents a relatively fair situation in such an unfair contest. We conduct a brief discussion about the "fairness" condition in the following subsection, and as will be shown in later sections, this condition plays a very important role in the optimization of unfair contests.

2.3 The "Fairness" Condition

The condition $\theta_1 v_1 + \theta_2 = v_2$ represents the scenario when both players make effort levels equal to their valuations of the prize, and they receive the same score in this unfair contest. This scenario can be interpreted as a fair case in an unfair setting. Since this equation implies an equal chance of winning if both players are making highest reasonable effort, we refer to this from now on as the *fairness condition* for unfair contests. In a special case when the agents are symmetric (and their valuation is normalized to 1), the fairness condition becomes $\theta_1 + \theta_2 = 1$.

3 Optimal Contest with Maximized Total Effort

We now consider the contest designer's problem, by endogenizing the parameters $\theta_1$and θ_2 based on the designer's objective. Suppose that the designer would like to maximize the expected total effort level of the two players, which is a common objective in the contest literature. The motivation is that a contest designer may wannt to maximize the total productivity of the contestants (in the case where output is proportional to effort). The designer can choose any combination of additive bias (θ_2) and multiplicative bias (θ_1) to help achieve his objective. We would like to ask the following question: how should the designer set the levels of favors of different types across players who differ in their valuations?

3.1 Total Effort Maximization

Given the equilibrium strategies characterized in Theorem 1, we derive the optimal choices for the contest designer across all 4 cases described in Theorem 1 and then pin down the total expected effort maximizing optimal choice among the cases.

Case (i): $\theta_2 \leq 0$ and $\theta_1 v_1 + \theta_2 \geq v_2$:

The designer's objective is to maximize the following expression:

$$E[x_1 + x_2] = \frac{v_2 - 2\theta_2}{2\theta_1} + \frac{v_2^2}{2v_1\theta_1} \tag{1}$$

Taking the first order condition with respect to θ_2, we have

$$\frac{\partial E[x_1 + x_2]}{\partial \theta_2} = -\frac{1}{\theta_1} < 0 \tag{2}$$

It is easy to see from the above condition that the designer should set the additive bias θ_2 as low as possible. Note that in this case the range of parameters are $v_2 - v_1\theta_1 \leq \theta_2 < 0$ and $\theta_1 > \frac{v_2}{v_1}$. Because the second equation is always valid regardless of parameters, thus we let $\theta_2 = v_2 - v_1\theta_1$. It is worth noting that by setting $\theta_2 = v_2 - v_1\theta_1$, this means exactly the fairness condition $\theta_1 v_1 + \theta_2 = v_2$. In other words, the fairness condition is a necessary condition for the optimization in Case (i).

Under this fairness condition, the optimization problem becomes

$$E[x_1 + x_2] = v_1 + \frac{v_2^2 - v_1 v_2}{2v_1\theta_1} \tag{3}$$

Taking the first order condition with respect to θ_1, we have

$$\frac{\partial E[x_1 + x_2]}{\partial \theta_1} = \frac{v_1 v_2 - v_2^2}{2v_1\theta_1^2} > 0 \tag{4}$$

It is easy to see from the above condition that the designer should set the additive bias θ_1 as high as possible. Since $\theta_1 \in [\underline{\theta}_1, \bar{\theta}_1]$, let $\theta_1 = \bar{\theta}_1 > 1$. We have $\theta_2 = v_2 - v_1\bar{\theta}_1$, and $E[x_1 + x_2] = v_1 + \frac{v_2^2 - v_1 v_2}{2v_1\bar{\theta}_1}$.

Case (ii): $\theta_2 \leq 0$ **and** $0 < \theta_1 v_1 + \theta_2 \leq v_2$**:**

The designer's objective is to maximize the following expression:

$$E[x_1 + x_2] = \theta_2 + \frac{v_1\theta_1}{2v_2}(v_1 + v_2) + \frac{\theta_2^2}{2v_1 v_2\theta_1}(v_2 - v_1) \tag{5}$$

Taking the first order condition with respect to θ_2, we have

$$\frac{\partial E[x_1 + x_2]}{\partial \theta_2} = 1 + \frac{2\theta_2}{2v_1 v_2\theta_1}(v_2 - v_1) > 0 \tag{6}$$

It is easy to see from the above condition that the designer should set the additive bias θ_2 as high as possible. Note that in this case the range of parameters are $-v_1\theta_1 < \theta_2 \leq v_2 - v_1\theta_1$. Thus we let $\theta_2 = v_2 - v_1\theta_1$. Again this means exactly the fairness condition $\theta_1 v_1 + \theta_2 = v_2$.

Under this fairness condition, the optimization problem becomes

$$E[x_1 + x_2] = v_1 + \frac{v_2^2 - v_1 v_2}{2v_1\theta_1} \tag{7}$$

It is easy to see the above expression is exactly the same as that in Case (i). So similarly we have $\theta_1 = \bar{\theta}_1 > 1$, $\theta_2 = v_2 - v_1\bar{\theta}_1$, and $E[x_1 + x_2] = v_1 + \frac{v_2^2 - v_1 v_2}{2v_1\bar{\theta}_1}$, with the results the same as in Case (ii).

Case (iii): $0 \leq \theta_2 < v_2$ **and** $\theta_1 v_1 + \theta_2 \geq v_2$**:**

The designer's objective is to maximize the following expression:

$$E[x_1 + x_2] = \frac{(v_2 - \theta_2)^2}{2v_2\theta_1} + \frac{v_2^2 - \theta_2^2}{2v_1\theta_1} \tag{8}$$

Taking the first order condition with respect to θ_2, we have

$$\frac{\partial E[x_1 + x_2]}{\partial \theta_2} = \frac{\theta_2 - v_2}{v_2\theta_1} - \frac{\theta_2}{v_1\theta_1} < 0 \tag{9}$$

It is easy to see from the above condition that the designer should set the additive bias θ_2 as low as possible. Note that in this case the range of parameters are $0 \le \theta_2 < v_2$ and $v_2 - v_1\theta_1 \le \theta_2$. Thus we let $\theta_2 = \max\{0, v_2 - v_1\theta_1\}$.

The optimization problem now becomes

$$E[x_1 + x_2] = \begin{cases} v_2 + \frac{v_1^2\theta_1}{2v_2} - \frac{v_1^2\theta_1}{2v_1} & \text{if } v_2 - v_1\theta_1 \ge 0 \\ \frac{v_2^2}{2v_2\theta_1} + \frac{v_2^2}{2v_1\theta_1} & \text{if } v_2 - v_1\theta_1 \le 0 \end{cases} \tag{10}$$

Taking the first order condition with respect to θ_1, we have

$$\frac{\partial E[x_1 + x_2]}{\partial \theta_1} = \begin{cases} \frac{v_1^2}{2v_2} - \frac{v_1^2}{2v_1} > 0 & \text{if } v_2 - v_1\theta_1 \ge 0 \\ -\frac{v_2^2}{2v_2\theta_1^2} - \frac{v_2^2}{2v_1\theta_1^2} < 0 & \text{if } v_2 - v_1\theta_1 \le 0 \end{cases} \tag{11}$$

No matter whether $v_2 - v_1\theta_1 \ge 0$ or $v_2 - v_1\theta_1 \le 0$, θ_1 should be set at $\theta_1 = \frac{v_2}{v_1}$. Thus we have $\theta_2 = 0$, and $E[x_1 + x_2] = \frac{v_1+v_2}{2}$. It is worth noting that in Case (iii), the fairness condition still holds.

Case (iv): $0 \le \theta_2 < v_2$ **and** $0 < \theta_1 v_1 + \theta_2 \le v_2$**:**

The designer's objective is to maximize the following expression:

$$E[x_1 + x_2] = \frac{v_1^2\theta_1}{2v_2} + \frac{v_1\theta_1}{2} + \theta_2 \tag{12}$$

Taking the first order condition with respect to θ_2, we have

$$\frac{\partial E[x_1 + x_2]}{\partial \theta_2} = 1 > 0 \tag{13}$$

It is easy to see from the above condition that the designer should set the additive bias θ_2 as high as possible. Note that in this case the range of parameters are $0 \le \theta_2 < v_2$ and $\theta_2 \le v_2 - v_1\theta_1$. Thus we let $\theta_2 = v_2 - v_1\theta_1$, which again implies the fairness condition.

The optimization problem now becomes

$$E[x_1 + x_2] = \frac{v_1^2\theta_1}{2v_2} + \frac{v_1\theta_1}{2} + v_2 - v_1\theta_1 \tag{14}$$

Taking the first order condition with respect to θ_1, we have

$$\frac{\partial E[x_1 + x_2]}{\partial \theta_1} = \frac{v_1^2}{2v_2} - \frac{v_1}{2} > 0 \tag{15}$$

It is easy to see from the above condition that the designer should set the additive bias θ_1 as high as possible. Since $\theta_1 \leq \frac{v_2-\theta_2}{v_1} \leq \frac{v_2}{v_1}$, let $\theta_1 = \frac{v_2}{v_1}$. Thus we have $\theta_2 = 0$, and $E[x_1 + x_2] = \frac{v_1+v_2}{2}$.

Among all the 4 cases, there are two different results: One is $\theta_1 = \bar{\theta}_1$, $\theta_2 = v_2 - v_1\bar{\theta}_1$, and $E[x_1 + x_2] = v_1 + \frac{v_2^2 - v_1 v_2}{2\bar{\theta}_1 v_1}$ (Cases (i) and (ii)), and the other is $\theta_1 = \frac{v_2}{v_1}$, $\theta_2 = 0$, and $E[x_1 + x_2] = \frac{v_1+v_2}{2}$ (Cases (iii) and (iv)). Since $v_1 > v_2$, we have $v_1 + \frac{v_2^2 - v_1 v_2}{2\bar{\theta}_1 v_1} > v_1 + \frac{v_2^2 - v_1 v_2}{2v_1} > v_1 + \frac{v_2 - v_1}{2} = \frac{v_1+v_2}{2}$. Therefore, the optimal choice for the contest designer is $(\theta_1^*, \theta_2^*) = \left(\bar{\theta}_1, v_2 - v_1\bar{\theta}_1\right)$, with the optimized expected total effort level $v_1 + \frac{v_2^2 - v_1 v_2}{2\bar{\theta}_1 v_1}$. We summarize this result in the following proposition.

Proposition 2 *A Total-Effort Maximizing Designer should set* $(\theta_1^*, \theta_2^*) = \left(\bar{\theta}_1, v_2 - v_1\bar{\theta}_1\right)$ *for the unfair contest and the maximized total effort level is* $E^*[x_1 + x_2] = v_1 + \frac{v_2^2 - v_1 v_2}{2\bar{\theta}_1 v_1}$.

In other words, the designer who would like to maximize contestants' total effort will apply the maximum possible multiplicative bias to the more productive contestant, and provide an additive subsidy to the less productive contestant, such that their chances of winning the contest are equalized.

3.2 Remarks

1. First, note that the fairness condition $\theta_1 v_1 + \theta_2 = v_2$ is a necessary condition for the optimization of a designer who would like to maximize the total effort. In the next section, we will be able to show that this fairness condition is robust to various objectives of the designer. In Section 5, we will introduce a general model where the favoritism does not necessarily take the linear form and show that a generalized fairness condition still holds under such a setup.

2. Second, it is worth mentioning that the optimal favoritism plan is such that both dimensions of favoritism are utilized. This implies that the result in Li and Yu

(2012) that the additive bias dominates the multiplicative bias is only a special case under the condition where only one dimension of favoritism is allowed by the designer. When both dimensions of favoritism are feasible, the optimal contest will generate higher expected total effort than those contests where only one dimension of favoritism is utilized. To see that, we do the following comparisons. (2.1) Suppose that the designer can only adjust the multiplicative bias θ_1 and there is no additive bias ($\theta_2 = 0$). Based on the fairness condition $\theta_1 v_1 + \theta_2 = v_2$, we have $\theta_1 = \frac{v_2}{v_1}$, and hence $E^*[x_1 + x_2] = \frac{v_1+v_2}{2}$. This is exactly the result for the multiplicative model in Li & Yu (2012). (2.2) Suppose that the designer can only adjust the additive bias θ_2 and there is no multiplicative bias ($\theta_1 = 0$). Based on the fairness condition $\theta_1 v_1 + \theta_2 = v_2$, we have $\theta_2 = v_2 - v_1$, and hence $E^*[x_1 + x_2] = v_1 + \frac{v_2^2 - v_1 v_2}{2v_1}$. This is exactly the result for the additive model in Li & Yu (2012). The result for the additive model indeed dominates that for the multiplicative model since $v_1 + \frac{v_2^2 - v_1 v_2}{2v_1} > \frac{v_1+v_2}{2}$. However, if we are allowed to set favoritism on both dimensions, we will have $E^*[x_1 + x_2] = v_1 + \frac{v_2^2 - v_1 v_2}{2\bar{\theta}_1 v_1}$, which is greater than $v_1 + \frac{v_2^2 - v_1 v_2}{2v_1}$.

3. Third, the maximized total effort level is increasing in $\bar{\theta}_1$, and as $\bar{\theta}_1 \to +\infty$, the maximized total effort level is approaching v_1.

4. Finally, in the symmetric case where $v_1 = v_2 = v$ and v is normalized to 1, we have the two biases $(\theta_1^*, \theta_2^*) = \left(\bar{\theta}_1, 1 - \bar{\theta}_1\right)$ and the maximized total effort level 1.

3.3 Optimal Contest under Alternative Objectives

In this section we allow the contest designer to pursue different objectives: maximizing the total scores, maximizing the highest effort, and maximizing the winning effort.

3.4 Maximizing the Total Scores

Sometimes the designer may have an objective to maximize the total score rather than the total effort. Examples of this include scenarios where public opinion on the total output of a contest matters more to the designer than the surplus obtained from contestants' efforts.

Given the equilibrium strategies characterized in Theorem 1, we once again derive the optimal choices for the contest designer across all 4 cases and then determine the score maximizing one among the cases.

Case (i): $\theta_2 \leq 0$ **and** $\theta_1 v_1 + \theta_2 \geq v_2$**:**

The designer's objective is to maximize the following expression:

$$E[s_1 + s_2] = \frac{v_2}{2} + \frac{v_2^2}{2v_1\theta_1} \tag{16}$$

Taking the first order condition with respect to θ_1, we have

$$\frac{\partial E[s_1 + s_2]}{\partial \theta_1} = -\frac{v_2^2}{2v_1\theta_1^2} < 0 \tag{17}$$

It is easy to see from the above condition that the designer should set the additive bias θ_1 as low as possible. Note that in this case the range of parameters are $v_2 - v_1\theta_1 \leq \theta_2 < 0$ and $\theta_1 > \frac{v_2}{v_1}$. Thus we let $\theta_1 = \frac{v_2}{v_1}$ and can have $E[s_1 + s_2] = v_2$.

Case (ii): $\theta_2 \leq 0$ **and** $0 < \theta_1 v_1 + \theta_2 \leq v_2$**:**

The designer's objective is to maximize the following expression:

$$E[s_1 + s_2] = \frac{(v_1\theta_1 + \theta_2)^2}{2v_2} + \frac{(v_1\theta_1 + \theta_2)^2}{2v_1\theta_1} \tag{18}$$

Simply, when $v_1\theta_1 + \theta_2 = v_2$, and $\theta_1 = \frac{v_2}{v_1}$, the maximum total score for the designer is also v_2.

Case (iii): $0 \leq \theta_2 < v_2$ **and** $\theta_1 v_1 + \theta_2 \geq v_2$**:**

The designer's objective is to maximize the following expression:

$$E[s_1 + s_2] = \frac{v_2^2 - \theta_2^2}{2v_2} + \frac{v_2^2 - \theta_2^2}{2v_1\theta_1} \tag{19}$$

Taking the first order condition with respect to θ_2, we have

$$\frac{\partial E[s_1 + s_2]}{\partial \theta_2} = -\frac{\theta_2}{v_2} - \frac{\theta_2}{v_1\theta_1} < 0 \tag{20}$$

Similarly, we let $\theta_2 = v_2 - v_1\theta_1$, and the optimization problem becomes

$$E[s_1 + s_2] = \frac{v_2^2 - (v_2 - v_1\theta_1)^2}{2v_2} + \frac{v_2^2 - (v_2 - v_1\theta_1)^2}{2v_1\theta_1} \tag{21}$$

Taking the first order condition with respect to θ_1, we have

$$\frac{\partial E[s_1 + s_2]}{\partial \theta_1} = \frac{v_1}{2} - \frac{v_1\theta_1^2}{v_2} \tag{22}$$

Thus, when $\theta_1 = \frac{v_2}{2v_1}$, the designer gets a maximum value of total score $\frac{9v_2}{8}$.

Case (iv): $0 \le \theta_2 < v_2$ **and** $0 < \theta_1 v_1 + \theta_2 \le v_2$**:**
The designer's objective is to maximize the following expression:

$$E[s_1 + s_2] = (\frac{\theta_1 v_1 + v_2}{2v_2})(2\theta_2 + v_1\theta_1) \tag{23}$$

Taking the first order condition with respect to θ_2, we have

$$\frac{\partial E[s_1 + s_2]}{\partial \theta_2} > 0 \tag{24}$$

Similarly, we let $\theta_2 = v_2 - v_1\theta_1$, and the optimization problem becomes

$$E[s_1 + s_2] = v_2 + \frac{(v_2 - v_1\theta_1)\theta_1 v_1}{2v_2} \tag{25}$$

The expression above reaches its maximum value when $\theta_1 = \frac{v_2}{2v_1}$, and the maximum value of total score is $\frac{9v_2}{8}$.

Among all the 4 cases, there are two different results: One is $\theta_1 = \frac{v_2}{v_1}$, $\theta_2 = 0$, and $E[s_1 + s_2] = v_2$ (Cases (i) and (ii)), and the other is $\theta_1 = \frac{v_2}{2v_1}$, $\theta_2 = \frac{v_2}{2}$, and $E[s_1 + s_2] = \frac{9v_2}{8}$ (Cases (iii) and (iv)). Obviously $\frac{9v_2}{8} > v_2$. Therefore, the optimal choice for the contest designer is $(\theta_1^*, \theta_2^*) = (\frac{v_2}{2v_1}, \frac{v_2}{2})$, with the optimized expected total effort level $\frac{9v_2}{8}$. Note that in this case the fairness condition is also satisfied. We summarize this result in the following proposition.

Proposition 3 *A Total-Score Maximizing Designer should set* $(\theta_1^*, \theta_2^*) = (\frac{v_2}{2v_1}, \frac{v_2}{2})$ *for the unfair contest and the maximized total effort level is* $E^*[s_1 + s_2] = \frac{9v_2}{8}$

Compared to the case of maximizing total effort, the designer who wishes to maximize the total scores of contestants, will provide a multiplicative favor to the less capable contestant, and applies an additive favor to the more capable contestant. Here, the designer is not focused on maximizing contestants' efforts, but on equalizing their scores directly.

3.5 Maximizing the Highest Effort

One can think of many situations in which the contest designer will only care about the most productive contestant. For example, any contest in which the designer can only make use of the most efficient entry in the contest, has this feature. As in the previous objectives, we analyze the optimal design problem, given players' equilibrium strategies.

Given the equilibrium strategies characterized in Theorem 1, we derive the optimal choices for the contest designer across all 4 cases.

Case (i): $\theta_2 \leq 0$ **and** $\theta_1 v_1 + \theta_2 \geq v_2$**:**

Subcase (i.1): $\theta_1 v_2 > v_2 - \theta_2$:

In this subcase, we have

$$\begin{aligned}
E[max(x_1, x_2)] &= E[x_1|x_1 \geq x_2] \cdot P(x_1 \geq x_2) + E[x_2|x_1 < x_2] \cdot P(x_1 < x_2) \\
&= \int_{-\frac{\theta_2}{\theta_1}}^{\frac{v_2-\theta_2}{\theta_1}} x \cdot \frac{\theta_1}{v_2}\left(\frac{x}{v_1\theta_1} + \frac{v_1\theta_1 - v_2}{v_1\theta_1}\right) dx + \int_{-\frac{\theta_2}{\theta_1}}^{\frac{v_2-\theta_2}{\theta_1}} x \cdot \frac{1}{v_1\theta_1} \cdot \frac{\theta_1 x + \theta_2}{v_2} dx + \int_{\frac{v_2-\theta_2}{\theta_1}}^{v_2} \frac{x}{v_1\theta_1} dx \\
&= \frac{v_2^2}{6v_1\theta_1^3} - \frac{\theta_2^2}{2v_1\theta_1^3} + \frac{v_2}{2\theta_1} - \frac{v_2^2}{2v_1\theta_1^2} - \frac{\theta_2}{\theta_1} + \frac{v_2\theta_2}{v_1\theta_1^2} + \frac{v_2^2}{2v_1\theta_1}
\end{aligned}$$

Taking the first order condition with respect to θ_2, we have

$$\frac{\partial E}{\partial \theta_2} = \frac{v_2\theta_1 - v_1\theta_1^2 - \theta_2}{v_1\theta_1^3} \tag{26}$$

Since $\theta_2 \geq v_2 - v_2\theta_1$, we have $\frac{\partial E}{\partial \theta_2} \leq \frac{v_2\theta_1 - v_1\theta_1^2 + v_2 - v_2\theta_1}{v_1\theta_1^3} = \frac{v_2 - v_1\theta_1^2}{v_1\theta_1^3}$. Furthermore, $\theta_1 > v_2/v_1$ gives us $\frac{\partial E}{\partial \theta_2} < 0$.

Therefore, we should have $v_2\theta_1 + \theta_2 = v_2$. This is again the fairness condition.

For the symmetric case, i.e. $v_1 = v_2 = 1$, $\theta_2 > 1 - \theta_1$, we have

$$\frac{\partial E}{\partial \theta_2} = \frac{\theta_1(1-\theta_1) - \theta_2}{\theta_1^3} < \frac{-(1-\theta_1)^2}{\theta_1^3} < 0 \tag{27}$$

This indicates that for a symmetric case a fair design is optimal. Letting $\theta_2 = v_2 - v_1\theta_1$, the optimization problem becomes

$$E = 1 + \frac{1}{3\theta_1^3} + \frac{3}{2\theta_1^2} - \frac{3}{2\theta_1} \tag{28}$$

Taking the first order condition with respect to θ_1, we have

$$\frac{\partial E}{\partial \theta_1} = \frac{3(\theta_1-1)^2+1}{4\theta_1^4} > 0 \tag{29}$$

Therefore, when $\theta_1 = \bar{\theta}_1$, the designer could get a maximum value of highest effort $1+\frac{1}{3\bar{\theta}_1^3}+\frac{3}{2\bar{\theta}_1^2}-\frac{3}{2\bar{\theta}_1}$ in the symmetric case.

Subcase (i.2): $\theta_1 v_2 \le v_2 - \theta_2$:

In this subcase, we have

$$\begin{aligned} E[max(x_1,x_2)] &= E[x_1|x_1 \ge x_2]\cdot P(x_1 \ge x_2) + E[x_2|x_1 < x_2]\cdot P(x_1 < x_2) \\ &= \textstyle\int_{-\frac{\theta_2}{\theta_1}}^{v_2} x\cdot\frac{\theta_1}{v_2}\left(\frac{x}{v_1\theta_1}+\frac{v_1\theta_1-v_2}{v_1\theta_1}\right)dx + \int_{v_2}^{\frac{v_2-\theta_2}{\theta_1}}\frac{x\theta_1}{v_2}dx + \int_{-\frac{\theta_2}{\theta_1}}^{v_2} x\cdot\frac{1}{v_1\theta_1}\cdot\frac{\theta_1 x+\theta_2}{v_2}dx \end{aligned}$$

Taking the first order condition with respect to θ_2, we have

$$\begin{aligned} \tfrac{\partial E}{\partial \theta_2} &= \tfrac{\theta_2^2}{2v_1v_2\theta_1^3}+\tfrac{\theta_2}{v_1\theta_1^2}-\tfrac{1}{\theta_1}+\tfrac{v_2}{2v_1\theta_1} \\ &= \tfrac{(\theta_2+v_2\theta_1)^2-2v_1v_2\theta_1^2}{2v_1v_2\theta_1^3} \\ &< \tfrac{v_2-2v_1\theta_1^2}{2v_1\theta_1^3} < 0 \end{aligned}$$

Since $\theta_2 \in (v_2-v_1\theta_1, v_2-v_2\theta_1)$, the maximum value is reached when $\theta_2 = v_2 - v_1\theta_2$. This indicates that the fair design is optimal in this subcase.

Case (ii): $\theta_2 \le 0$ and $0 < \theta_1 v_1 + \theta_2 \le v_2$:

The designer's objective is to maximize the following expression:

$$\begin{aligned} E[max(x_1,x_2)] = &\int_{-\frac{\theta_2}{\theta_1}}^{\theta_1 v_1+\theta_2} x\cdot\frac{\theta_1}{v_2}\left(\frac{x}{v_1\theta_1}-\frac{\theta_2}{v_1\theta_1}\right)dx + \int_{\theta_1 v_1+\theta_2}^{v_1}\frac{x\theta_1}{v_2}dx \\ &+\int_0^{-\frac{\theta_2}{\theta_1}}\frac{x}{v_1\theta_1}\left(1-\frac{\theta_2+\theta_1 v_1}{v_2}\right)dx + \int_{-\frac{\theta_2}{\theta_1}}^{\theta_1 v_1+\theta_2} x\cdot\frac{1}{v_1\theta_1}\cdot\left[\frac{\theta_1 x}{v_2}+1-\frac{\theta_1 v_1}{v_2}\right]dx \end{aligned} \tag{30}$$

Taking the partial derivatives with respect to θ_2, we have

$$\begin{aligned} \frac{\partial E}{\partial \theta_2} = &\frac{3}{2v_1v_2}(v_1\theta_1+v_2)^2+\frac{\theta_2^2}{2v_1v_2\theta_1^3}+1+\frac{\theta_2}{v_1\theta_1}-\frac{2\theta_1\theta_2}{v_2}- \\ &\frac{v_1\theta_1(\theta_1+1)}{v_2}-\frac{\theta_2}{v_2}-\frac{\theta_2^2}{v_1v_2}-\frac{2\theta_2}{v_2\theta_1^2}+\frac{3\theta_2^2}{2v_1v_2\theta_1^2} \end{aligned}$$

Rearranging the terms in the above expression, we obtain

$$\frac{\partial E}{\partial \theta_2} = \frac{3}{2v_1v_2}(v_1\theta_1+v_2)^2 - \frac{2\theta_1\theta_2}{v_2} - \frac{v_1\theta_1(\theta_1+1)}{v_2} - \frac{\theta_2^2}{v_1v_2} + \frac{\theta_2^2}{2v_1v_2\theta_1^3} + (1+\frac{\theta_2}{v_1\theta_1}) - \frac{\theta_2}{v_2} - \frac{2\theta_2}{v_2\theta_1^2} + \frac{3\theta_2^2}{2v_1v_2\theta_1^2}$$

Note that

$$\frac{\partial E}{\partial \theta_2} > \frac{(\theta_1+\theta_2)^2}{2v_1v_2} + \frac{\theta_2^2}{2v_1v_2\theta_1^3} + (1+\frac{\theta_2}{v_1\theta_1}) - \frac{\theta_2}{v_2} - \frac{2\theta_2}{v_2\theta_1^2} + \frac{3\theta_2^2}{2v_1v_2\theta_1^2} > 0 \tag{31}$$

Therefore, in this case we should have $\theta_1 v_1 + \theta_2 = v_2$, which again implies that a fair design is optimal.

Case (iii): $0 \leq \theta_2 < v_2$ **and** $\theta_1 v_1 + \theta_2 \geq v_2$**:**

Subcase (iii.1): $\theta_1 v_2 > v_2 - \theta_2$:

In this subcase, we have

$$\begin{aligned} E[max(x_1,x_2)] = & \int_0^{\theta_2} x \cdot \frac{\theta_1}{v_2} \frac{v_1\theta_1+\theta_2-v_2}{v_1\theta_1} dx + \int_{\theta_2}^{\frac{v_2-\theta_2}{\theta_1}} x \cdot \frac{\theta_1}{v_2} \frac{x+v_1\theta_1-v_2}{v_1\theta_1} dx \\ & + \int_{\theta_2}^{\frac{v_2-\theta_2}{\theta_1}} \frac{x}{v_1\theta_1} \cdot \frac{\theta_2+\theta_1 x}{v_2} dx + \int_{\frac{v_2-\theta_2}{\theta_1}}^{v_2} \frac{x}{v_1\theta_1} dx \end{aligned} \tag{32}$$

Considering the symmetric case, we have

$$\frac{\partial E}{\partial \theta_2} = -\frac{\theta_2^2}{2} + \frac{-2\theta_2^2-(\theta_2-1)^2-1}{2\theta_1} + \frac{1-\theta_2}{\theta_1^2} + \frac{-(1+\theta_2)^2-2\theta_2^2}{2\theta_1^3} \tag{33}$$

Since $1-\theta_2 < \theta_1$, note that

$$\frac{\partial E}{\partial \theta_2} < -\frac{\theta_2^2}{2} + \frac{-2\theta_2^2-(\theta_2-1)^2-1}{2\theta_1} + \frac{1}{\theta_1} + \frac{-(1+\theta_2)^2-2\theta_2^2}{2\theta_1^3} < 0 \tag{34}$$

Therefore, in a symmetric case, we should have $\theta_1 v_1 + \theta_2 = v_2$, which again implies that a fair design is optimal.

Subcase (iii.2): $\theta_1 v_2 < v_2 - \theta_2$:

In this subcase, we have

$$E[max(x_1,x_2)] = \int_0^{\theta_2} x\cdot\frac{\theta_1}{v_2}\frac{v_1\theta_1+\theta_2-v_2}{v_1\theta_1}dx + \int_{\theta_2}^{v_2} x\cdot\frac{\theta_1}{v_2}\frac{x+v_1\theta_1-v_2}{v_1\theta_1}dx$$
$$+\int_{v_2}^{\frac{v_2-\theta_2}{\theta_1}} \frac{\theta_1 x}{v_2}dx + \int_{\theta_2}^{v_2}\frac{x}{v_1\theta_1}\cdot\frac{\theta_2+\theta_1 x}{v_2}dx \tag{35}$$

Taking the partial derivatives with respect to θ_2, we have

$$\frac{\partial E}{\partial\theta_2} = -\frac{\theta_2^2}{2v_1v_2} - \frac{1}{\theta_1} + \frac{\theta_2}{v_2\theta_1} + \frac{v_2}{2v_1\theta_1} - \frac{3\theta_2^2}{2v_1v_2\theta_1} \tag{36}$$

Note that

$$\frac{\partial E}{\partial\theta_2} < -\frac{\theta_2^2}{2v_1v_2} + \frac{v_2}{2v_1\theta_1} - \frac{3\theta_2^2}{2v_1v_2\theta_1} < 0 \tag{37}$$

Therefore, a fair design is also optimal in this case.

Case (iv): $0 \le \theta_2 < v_2$ and $0 < \theta_1 v_1 + \theta_2 \le v_2$:

The designer's objective is to maximize the following expression:

$$E[max(x_1,x_2)] = \int_{\theta_2}^{v_1\theta_1+\theta_2} x\cdot\frac{\theta_1}{v_2}\cdot\frac{x-\theta_2}{v_1\theta_1}dx + \int_{v_1\theta_1+\theta_2}^{v_1} x\cdot\frac{\theta_1}{v_2}dx$$
$$+\int_{\theta_2}^{v_1\theta_1+\theta_2}\frac{x}{v_1\theta_1}\cdot\frac{v_2-v_1\theta_1+\theta_1 x}{v_2}dx \tag{38}$$

Taking the first order condition with respect to θ_2, we have

$$\frac{\partial E}{\partial\theta_2} = \frac{v_1\theta_1^2}{2v_2} + \frac{\theta_1\theta_2}{v_2} + \frac{v_2-v_1\theta_1}{v_2} > 0 \tag{39}$$

Thus, the fairness condition should again be satisfied in the optimal case.

Among all the 4 cases, for the contests with symmetric agents the fairness condition is always satisfied for the optimization. For the general contests with asymmetric agents, in most cases (except Subcase (iii.1)), we also have the fairness condition necessary for the optimization of the highest effort. However, we are not sure about the Subcase (iii.1) yet and further work is still needed.

3.6 Maximizing the Winning Effort

In other cases, the contest designer may not care as much about the highest effort level implemented in the contest, but may care about maximizing the winning effort level. Given the equilibrium strategies characterized in Theorem 1, we derive the optimal choices for the contest designer across all 4 cases and then choose the best one among the cases.

Case (i): $\theta_2 \leq 0$ **and** $\theta_1 v_1 + \theta_2 \geq v_2$**:**

The designer's objective is to maximize the following expression:

$$\begin{aligned}
E[win(x_1, x_2)] &= E[x_1|s_1 \geq s_2] \cdot P(s_1 \geq s_2) + E[x_2|s_1 < s_2] \cdot P(s_1 < s_2) \\
&= \int_{-\frac{\theta_2}{\theta_1}}^{\frac{v_2-\theta_2}{\theta_1}} x \cdot \tfrac{\theta_2}{v_2} \cdot F(\theta_1 x + \theta_2)dx + \int_0^{v_2} x \cdot \tfrac{1}{v_1\theta_1} \cdot F(\tfrac{x-\theta_2}{\theta_1}) \\
&= \tfrac{\theta_1}{2v_2} \cdot \left[\tfrac{v_1\theta_1+\theta_2-v_2}{v_1\theta_1}\right] \cdot \left[\tfrac{v_2^2-2v_2\theta_2}{\theta_1^2}\right] + \tfrac{\theta_1}{3v_1v_2}\left[\tfrac{(v_2-\theta_2)^3}{\theta_1^3} + \tfrac{\theta_2^3}{\theta_1^3}\right] + \tfrac{v_2^2}{3v_1\theta_1}
\end{aligned}$$

Taking the first order condition with respect to θ_2, we have

$$\frac{\partial E}{\partial \theta_2} = \frac{v_2}{2v_1\theta_1^2} - \frac{1}{\theta_1} < 0 \tag{40}$$

Therefore, $\theta_2 = v_2 - v_1\theta_1$, which means that fair design is optimal.

Letting $\theta_2 = v_2 - v_1\theta_1$, the optimization problem becomes

$$E = v_1 - \frac{v_2}{\theta_1} + \frac{v_2^2}{3v_1\theta_1^2} + \frac{v_2^2}{3v_1\theta_1} \tag{41}$$

Assuming $\theta_1 \geq \frac{v_2}{v_1}$, note that $v_1 > v_2$ implies that $v_1\theta_1 - v_2\theta_1 > 0$

$$\frac{\partial E}{\partial \theta_1} = \frac{v_2[3v_1\theta_1 - 2v_2 - v_2\theta_1]}{3v_1\theta_1^3} > \frac{v_2[2v_1\theta_1 - 2v_2]}{3v_1\theta_1^3} \geq 0 \tag{42}$$

The last inequality comes from the assumption $\theta_1 \geq \frac{v_2}{v_1}$.

Therefore, the maximum reached when $\theta_1 = \bar{\theta}_1$, the maximum value is $E[win(x_1, x_2)] = v_1 - \frac{v_2}{\bar{\theta}_1} + \frac{v_2^2}{3v_1\bar{\theta}_1^2} + \frac{v_2^2}{3v_1\bar{\theta}_1}$

Case (ii): $\theta_2 \leq 0$ **and** $0 < \theta_1 v_1 + \theta_2 \leq v_2$**:**

The designer's objective is to maximize the following expression:

$$
\begin{aligned}
E[win(x_1,x_2)] &= E[x_1|s_1 \geq s_2]\cdot P(s_1 \geq s_2) + E[x_2|s_1 < s_2]\cdot P(s_1 < s_2) \\
&= \tfrac{\theta_1 v_1^2}{3v_2} + \tfrac{\theta_2^3}{3v_1v_2\theta_1^2} + \tfrac{(v_1\theta_1+\theta_2)^2}{2v_1\theta_1} - \tfrac{(v_1\theta_1+\theta_2)^3}{6v_1v_2\theta_1}
\end{aligned}
$$

Taking the first order condition with respect to θ_2, we have

$$
\frac{\partial E}{\partial \theta_2} = \frac{\theta_2^2}{v_1v_2\theta_1^2} + \frac{(v_1\theta_1+\theta_2)(2v_2 - v_1\theta_1 - \theta_2)}{2v_1v_2\theta_1} > 0 \tag{43}
$$

Therefore, $\theta_2 = v_2 - v_1\theta_1$, which satisfies the fairness condition.

Letting $\theta_2 = v_2 - v_1\theta_1$, the optimization problem becomes

$$
E = v_1 + \frac{v_2^2}{3v_1\theta_1} + \frac{v_2^2}{3v_1\theta_1} - \frac{v_2}{\theta_1} \tag{44}
$$

Taking the first order condition with respect to θ_1, we have

$$
\frac{\partial E}{\partial \theta_1} = \frac{3v_1v_2\theta_1 - 2v_2^2 - v_2^2\theta_1}{3v_1\theta_1^3} \tag{45}
$$

The equation here is exactly the same as in Case (i). If we assume that θ_1 satisfies the same condition as Case (1): $\theta_1 \geq \frac{v_2}{v_1}$,we could have the same result here.

Case (iii): $0 \leq \theta_2 < v_2$ and $\theta_1 v_1 + \theta_2 \geq v_2$:

The designer's objective is to maximize the following expression:

$$
\begin{aligned}
E[win(x_1,x_2)] &= E[x_1|s_1 \geq s_2]\cdot P(s_1 \geq s_2) + E[x_2|s_1 < s_2]\cdot P(s_1 < s_2) \\
&= \tfrac{(v_2-\theta_2)^3}{3v_1v_2\theta_1^2} + \tfrac{\theta_1}{2v_2}(1 + \tfrac{\theta_2 - v_2}{v_1\theta_1})\tfrac{(v_2-\theta_2)^2}{\theta_1^2} + \tfrac{v_2^3-\theta_2^3}{3v_1v_2\theta_1} \\
&= \tfrac{(v_2-\theta_2)^2[3v_1\theta_1+\theta_2-v_2]}{6v_1v_2\theta_1^2} + \tfrac{v_2^3-\theta_2^3}{3v_1v_2\theta_1}
\end{aligned}
$$

Taking the first order condition with respect to θ_1, we have

$$
\frac{\partial E}{\partial \theta_1} = -\frac{(v_2-\theta_2)^2[3v_1\theta_1 + 2\theta_2 - 2v_2]}{6v_1v_2\theta_1^3} - \frac{v_2^3 - \theta_2^3}{3v_1v_2\theta_1^2} < 0 \tag{46}
$$

Therefore, we should have $\theta_1 = \frac{v_2-\theta_2}{v_1}$, which is exactly the fairness condition.

Letting $\theta_1 = \frac{v_2-\theta_2}{v_1}$, the optimization problem becomes

$$
E = \frac{v_1+v_2}{3} + \frac{\theta_2 v_2 + \theta_2^2 - v_1\theta_2}{3v_2} \tag{47}
$$

Replacing θ_2 by $v_2 - v_1\theta_1$, we could get the same result as Case (iv):

$$E = \frac{v_1^2\theta_1}{3v_2} + v_2 - v_1\theta_1 + \frac{v_1^2\theta_1^2}{3v_2} \tag{48}$$

Now consider equation (**??**), since E is a function of $\theta_2 \in (0, v_2)$, the maximum value of the winning effort is $E(\theta_2 = 0) = \frac{v_1+v_2}{3}$ if $v_1 \geq 2v_2$ or $E(\theta_2 = v_2) = v_2$ if $2v_2 > v_1 > v_2$.

Case (iv): $0 \leq \theta_2 < v_2$ **and** $0 < \theta_1 v_1 + \theta_2 \leq v_2$**:**

The designer's objective is to maximize the following expression:

$$\begin{aligned} E[win(x_1, x_2)] &= E[x_1|s_1 \geq s_2] \cdot P(s_1 \geq s_2) + E[x_2|s_1 < s_2] \cdot P(s_1 < s_2) \\ &= \tfrac{v_1^2\theta_1}{3v_2} + \tfrac{v_1\theta_1}{2} + \theta_2 + \tfrac{\theta_2^2}{2v_2} - \tfrac{(v_1\theta_1+\theta_2)^3}{6v_1v_2\theta_1} + \tfrac{\theta_2^3}{6v_1v_2\theta_1} \end{aligned}$$

Taking the first order condition with respect to θ_2, we have

$$\frac{\partial E}{\partial \theta_2} = \frac{2v_2 - v_1\theta_1}{2v_2} > 0 \tag{49}$$

Therefore, we should have $\theta_2 = v_2 - v_1\theta_1$, which is exactly the fairness condition.

Letting $\theta_2 = v_2 - v_1\theta_1$, the optimization problem becomes

$$E = \frac{v_1^2\theta_1}{3v_2} + v_2 - v_1\theta_1 + \frac{v_1^2\theta_1^2}{3v_2} \tag{50}$$

Since the above expression is the same as the equation (**??**) in Case (iii), the optimal result is also the same, that is, the maximum value of the winning effort is $E(\theta_2 = 0) = \frac{v_1+v_2}{3}$ if $v_1 > 2v_2$ or $E(\theta_2 = v_2) = v_2$ if $2v_2 > v_1 > v_2$.

Among all the 4 cases, there are two different results:

(1) One is $\theta_1 = \bar{\theta}_1$, $\theta_2 = v_2 - v_1\bar{\theta}_1$ and $E[win(x_1, x_2)] = v_1 - \frac{v_2}{\bar{\theta}_1} + \frac{v_2^2}{3v_1\bar{\theta}_1^2} + \frac{v_2^2}{3v_1\bar{\theta}_1}$ (Cases (i) and (ii)).

(2) The other is if $2v_2 > v_1 > v_2$, $\theta_1 = 0$, $\theta_2 = v_2$, and $E[win(x_1, x_2)] = v_2$; if $v_1 > 2v_2$, $\theta_1 = \frac{v_2}{v_1}$, $\theta_2 = 0$ and $E[win(x_1, x_2)] = \frac{v_1+v_2}{3}$ (Cases (iii) and (iv)).

To compare these two results, first we know the result in (1) is increasing with respect to θ_1, so the minimum of result (1) is E$[\theta_1 = v_2/v_1] = \frac{v_1+v_2}{3}$. This shows that when $\bar{\theta}_1 = \frac{v_2}{v_1}$, the payoff in result (2) weakly dominate result (1). Then as we increase $\bar{\theta}_1$, there will exist some crucial point say v_0, such that if $v_1 < v_0$, the second result is better, and if $v_1 \geq v_0$ the first. Eventually, when $\bar{\theta}_1$ exceeds some value θ_1^*, such that payoff in result (1) is always larger than result (2), we would say that (1) dominate (2) in this condition.

4 A Model with General Forms of Favoritism

Suppose that the contest designer can take a more general, non-linear form of favoritism by setting the score of player 1 as $f(b_1)$, while maintaining player 2's score as b_2. Regarding the functional form of favoritism, assume that (1) f is a bijective function; (2) f is increasing i.e. $f'(x) > 0$; (3) f and f^{-1} are integrable and continuous.

Thus, players' payoff functions are the following:

$$\begin{aligned} u_1 &= Pr(f(b_1) \geq b_2) \cdot v_1 - b_1 \\ &= F_2(f(b_1))v_1 - b_1 \\ u_2 &= Pr(f(b_1) \leq b_2) \cdot v_2 - b_2 \\ &= F_1(f^{-1}(b_2))v_2 - b_2 \end{aligned}$$

4.1 The Equilibrium Allocation

Here we characterize players' equilibrium strategies for the unfair contests with general forms of favoritism.

Theorem 2 *The all-pay-auction contest has the following mixed-strategy equilibrium* $(F_1(x_1), F_2(x_2))$ *for different environments of parameters* (v_1, v_2, f)*:*

(i) $f(0) \leq 0$ *and* $f(v_1) \geq v_2$*:* $F_1(x) = \begin{cases} 0 & \text{if } x \in [0, f^{-1}(0)) \\ \frac{f(x)}{v_2} & \text{if } x \in [f^{-1}(0), f^{-1}(v_2)) \\ 1 & \text{if } x \in [f^{-1}(v_2), +\infty) \end{cases}$, $F_2(x) = \begin{cases} 1 + \frac{f^{-1}(x) - f^{-1}(v_2)}{v_1} & \text{if } x \in [0, v_2) \\ 1 & \text{if } x \in [v_2, +\infty) \end{cases}$;

(ii) $f(0) \leq 0$ *and* $0 < f(v_1) \leq v_2$*:* $F_1(x) = \begin{cases} 1 - \frac{f(v_1)}{v_2} & \text{if } x \in [0, f^{-1}(0)) \\ 1 - \frac{f(v_1)}{v_2} + \frac{f(x)}{v_2} & \text{if } x \in [f^{-1}(0), v_1) \\ 1 & \text{if } x \in [v_1, +\infty) \end{cases}$,

$$F_2(x) = \begin{cases} \frac{f^{-1}(x)}{v_1} & x \in [0, f(v_1)) \\ 1 & x \in [f(v_1), +\infty) \end{cases};$$

(iii) $0 < f(0) < v_2$ *and* $f(v_1) \geq v_2$*:* $F_1(x) = \begin{cases} \frac{f(x)}{v_2} & \text{if } x \in [0, f^{-1}(v_2)) \\ 1 & \text{if } x \in [f^{-1}(v_2), +\infty) \end{cases}$, $F_2(x) =$

$$\begin{cases} 1 + \frac{-f^{-1}(v_2)}{v_1} & \text{if } x \in [0, f(0)) \\ 1 + \frac{f^{-1}(x) - f^{-1}(v_2)}{v_1} & \text{if } x \in [f(0), v_2) \\ 1 & \text{if } x \in [v_2, +\infty) \end{cases};$$

(iv) $0 < f(0) < v_2$ *and* $0 < f(v_1) \leq v_2$*:* $F_1(x) = \begin{cases} 1 + \frac{f(x) - f(v_1)}{v_2} & \text{if } x \in [0, v_1) \\ 1 & \text{if } x \in [v_1, +\infty) \end{cases}$,

$$F_2(x) = \begin{cases} 0 & \text{if } x \in [0, f(0)) \\ \frac{f^{-1}(x)}{v_1} & \text{if } x \in [f(0), f(v_1)) \\ 1 & \text{if } x \in [f(v_1), +\infty) \end{cases}.$$

Proof. See Appendix. ■

4.2 Total Effort Maximization

Given the equilibrium strategies characterized in Theorem 2, we derive the conditions under which the optimal choices can be made across all 4 cases.

Case (i): $f(0) \leq 0$ **and** $f(v_1) \geq v_2$**:**

The designer's objective is to maximize the following expression:

$$E[x_1 + x_2] = \int_{f^{-1}(0)}^{f^{-1}(v_2)} x \cdot \frac{f'(x)}{v_2} dx + \frac{f^{-1}(v_2) - f^{-1}(0)}{v_1} \int_0^{v_2} x \cdot \frac{(f^{-1}(x))'}{v_1} dx \tag{51}$$

Using integral by part, we have

$$E[x_1 + x_2] = f^{-1}(v_2) - \int_{f^{-1}(0)}^{f^{-1}(v_2)} \frac{f(x)}{v_2} dx + \frac{f^{-1}(v_2) - f^{-1}(0)}{v_1} \left[\frac{f^{-1}(v_2) v_2}{v_1} - \int_0^{v_2} \frac{f^{-1}(x)}{v_1} dx \right] \tag{52}$$

By the chain rule, we know

$$\frac{\partial E}{\partial f^{-1}(v_2)} = \frac{\partial E}{\partial v_2} \cdot f'(v_2) \tag{53}$$

Taking the first order condition with respect to v_2, we have

$$\frac{\partial E}{\partial v_2} = \frac{1}{v_2^2} \int_{[f^{-1}(0)]}^{f^{-1}(v_2)} f(x)dx + \frac{1}{v_1 f'(v_2)} \left[\frac{f^{-1} v_2}{v_1} - \int_0^{v_2} \frac{f^{-1}(x)}{v_1} dx \right] + \frac{f^{-1}(v_2) - f^{-1}(0)}{v_1} \frac{v_2}{f'(v_2) v_1} > 0 \tag{54}$$

This implies that $\frac{\partial E}{\partial f^{-1}(v_2)} > 0$ since $f'(x) > 0$.

Therefore, we have $f(v_1) = v_2$, which indicates that the fairness condition is necessary for optimality here.

Case (ii): $f(0) \leq 0$ **and** $0 < f(v_1) \leq v_2$**:**

The designer's objective is to maximize the following expression:

$$E[x_1 + x_2] = \frac{f(v_1)}{v_2} \int_{f^{-1}(0)}^{v_1} x \cdot \frac{f'(x)}{v_2} dx + \left(1 - \frac{f^{-1}(0)}{v_1}\right) \int_0^{f(v_1)} x \cdot \frac{(f^{-1}(x))'}{v_1} dx \tag{55}$$

Using integral by part, we have

$$E[x_1 + x_2] = \frac{f(v_1)}{v_2} \left[\frac{v_1 f(v_1)}{v_2} - \int_{f^{-1}(0)}^{v_1} \frac{f(x)}{v_2} dx \right] + f(v_1) - \int_0^{f(v_1)} \frac{f^{-1}(x)}{v_1} dx \tag{56}$$

From the chain rule we have

$$\frac{\partial E}{\partial f(v_1)} = \frac{\partial E}{\partial v_1} \cdot \frac{1}{f'(v_1)} \tag{57}$$

Taking the first order condition with respect to v_1, we have

$$\frac{\partial E}{\partial v_1} = \frac{2 v_1 f(v_1) f'(v_1)}{v_2^2} - \frac{f'(v_1)}{v_2} \int_{f^{-1}(0)}^{v_1} \frac{f(x)}{v_2} dx + \frac{1}{v_2^2} \int_0^{f(v_1)} f^{-1}(x) dx > 0 \tag{58}$$

This implies that $\frac{\partial E}{\partial f(v_1)} > 0$ since $f'(x) > 0$.

Therefore, we have $f(v_1) = v_2$, which indicates that the fairness condition is also satisfied.

Case (iii): $0 < f(0) < v_2$ **and** $f(v_1) \geq v_2$**:**

The designer's objective is to maximize the following expression:

$$E[x_1+x_2] = (1-\frac{f(0)}{v_2})\int_0^{f^{-1}(v_2)} x\cdot\frac{f'(x)}{v_2}dx + \frac{f^{-1}(v_2)}{v_1}\int_{f(0)}^{v_2} x\cdot\frac{(f^{-1}(x))'}{v_1}dx \tag{59}$$

Using integral by part, we have

$$E[x_1+x_2] = (1-\frac{f(0)}{v_2})\left[f^{-1}(v_2)-\int_0^{f^{-1}(v_2)}\frac{f(x)}{v_2}dx\right]+\frac{f^{-1}(v_2)}{v_1}\left[\frac{v_2 f^{-1}(v_2)}{v_1}-\int_{f(0)}^{v_2}\frac{f^{-1}(x)}{v_1}dx\right] \tag{60}$$

By the chain rule, we know

$$\frac{\partial E}{\partial f^{-1}(v_2)} = \frac{\partial E}{\partial v_2}\cdot f'(v_2) \tag{61}$$

Taking the first order condition with respect to v_2, we have

$$\frac{\partial E}{\partial v_2} = \frac{\int_0^{f^{-1}(v_2)} f(x)dx}{v_2^2} - \frac{f(0)f^{-1}(v_2)}{v_2^2} + \frac{2v_2 f^{-1}(v_2)}{v_1^2 f'(v_2)} - \frac{\int_{f(0)}^{v_2} f^{-1}(x)dx}{v_1^2 f'(v_2)} > 0 \tag{62}$$

This implies that $\frac{\partial E}{\partial f^{-1}(v_2)} > 0$ since $f'(x) > 0$.

Therefore, we have $f(v_1) = v_2$, which shows that the fairness condition is necessary in this case as well.

Case (iv): $0 < f(0) < v_2$ **and** $0 < f(v_1) \le v_2$**:**

The designer's objective is to maximize the following expression:

$$E[x_1+x_2] = \frac{f(v_1)-f(0)}{v_2}\int_0^{v_1} x\cdot\frac{f'(x)}{v_2}dx + \int_{f(0)}^{f(v_1)} x\cdot\frac{(f^{-1}(x))'}{v_1}dx \tag{63}$$

Using integral by part, we have

$$E[x_1+x_2] = \frac{f(v_1)-f(0)}{v_2}\left[\frac{v_1 f(v_1)}{v_2}-\int_0^{v_1}\frac{f(x)}{v_2}dx\right] + f(v_1) - \int_{f(0)}^{f(v_1)}\frac{f^{-1}(x)}{v_1}dx \tag{64}$$

By the chain rule, we know

$$\frac{\partial E}{\partial f(v_1)} = \frac{\partial E}{\partial v_1} \cdot \frac{1}{f'(v_1)} \tag{65}$$

Taking the first order condition with respect to v_1, we have

$$\frac{\partial E}{\partial v_1} = \frac{f'(v_1)f(v_1)v_1}{v_2^2} - \frac{f'(v_1)}{v_2^2}\int_0 v_1 f(x)dx + [f(v_1) - f(0)]\frac{f'(v_1)v_1}{v_2^2} + \frac{1}{v_1^2}\int_{f(0)}^{f(v_1)} f^{-1}(x)dx > 0 \tag{66}$$

This implies that $\frac{\partial E}{\partial f(v_1)} > 0$ since $f'(x) > 0$.

Therefore, we have $f(v_1) = v_2$, which implies that the fairness condition is satisfied.

It is easy to see that among all the 4 cases, the fairness condition is always satisfied for the optimization.

5 Conclusion

We consider an all-pay auction contest with two players who are asymmetric in their abilities in the contest. The contest designer can decide how to allocate favors to the players in different ways. Our benchmark model considers the case where the designer can allocate multiplicative and additive favors to each contestant, and shows that the optimal contest has the designer allocating a purely multiplicative favor to the more capable contestant, and allocating a purely additive favor or head start to the less able contestant. We further derive a "fairness condition" which the designer must satisfy under any of the following objectives: maximizing the total effort, maximizing the total score, maximizing the highest effort by a single player, or maximizing the winner's effort. Finally, we show that this fairness condition is robust to more generalized non-linear forms of favoritism.

We see several possible directions for future research. One potentially important direction is to consider a set-up with costly favors. Thus far, the favors in our model and in the literature are costless to the designer. However, such subsidies may incur a cost in the real world. Another possible direction is to generalize the model to the n-player case, instead of the two player setting examine here. Finally, we may consider the case where there is asymmetric information among players about the favors being given by the designer. We leave these studies for future work.

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