A Dynamic Model of Bertrand Competition for an Oligopolistic Market

Zeng Lian and Jie Zheng

Abstract We study an infinitely-repeated Bertrand competition game among a fixed number of firms in a market of both stochastic entry and stochastic demand. A firm's entry into market in the next period is possible by making a positioning investment with stochastic success rate. The market demand in the next period is also stochastic and will not be realized until the firm enters the market. A successful investment allows a firm to participate in the Bertrand competition and an unsuccessful investment prevents a firm from entering the market, for the next period. We characterize the symmetric Markov perfect Nash Equilibrium (SMPNE) of such a dynamic game, where a firm's strategy consists of two components: positioning strategy and pricing strategy. In examples with 1, 2, and 3 firms, we show the stage game market outcome, present the dynamic process of market structure, solve for the steady state of the dynamic system, and discuss about the speed of convergence to the steady state. Our work contributes to the dynamic oligopoly literature by allowing for two dimensions of stochastic uncertainty in firms' decision-making.

1 Introduction

It is well-observed that in the business world price wars usually persist in a dynamic pattern among firms producing similar products or firms providing similar services. Meanwhile, the market environment where the competition takes place seems to involve a lot of uncertainty. Thus, it is of theoretical importance and interest to study how firms compete and what the market equilibrium is in a dynamic environment with uncertainty.

In this research, we model the dynamic price competition among firms in a market with two types of uncertainty. The first type of uncertainty refers to firms' chance of entering and remaining in the market while the second type of uncertainty refers to the demand condition of the market. All firms may not participate in the market competition in any given period, and whether a firm can enter the market in a given period depends on whether it made a positioning investment in the previous period and whether such an investment

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Zeng Lian
International Business School, Beijing Foreign Studies University, Beijing, China, 100089, e-mail: lianzeng@bfsu.edu.cn

Jie Zheng
Department of Economics, School of Economics and Management, Tsinghua University, Beijing, China, 100084 e-mail: jie.academic@gmail.com

was successful, according to a stochastic binary distribution. In each period, the market demand can be either high or low, which is stochastically realized and positively correlated to the market demand condition in the previous period.

We characterize firms' strategies, which consist of both positioning strategy and pricing strategy, and focus on the symmetric Markov perfect Nash Equilibrium (SMPNE) of such a dynamic game. We present some examples for realizations of the dynamic patterns of market structure when firms use their SMPNE strategies, and we show that the realization patterns differ significantly, depending on the initial states. We also show the steady state distribution and obtain the expected duration for each state for the system with different number of firms. Furthermore, we investigate the speed of convergence to the steady state of the dynamic system.

The main contribution of our work is to allow for two dimensions of stochastic uncertainty in firms' decision-making under the setup of dynamic oligopoly. The dynamic game literature usually assumes that there is no uncertainty (Wang and Zheng, 2015) [22] or there is one type of uncertainty (Bloch, Eaton and Rothschild, 2014) [2]. In our setup, when firms make their positioning investment decisions, they need to take into account both the uncertainty of the result from the positioning investment and the uncertainty of the market demand condition given successful investment.

Over the last four decades, there have been a number of significant contributors to the industrial organization literature related to our paper, including Caves and Porter (1977) [4], Dixit (1979, 1980) [5, 6], Schmalensee (1978) [18], Spence (1977, 1978) [19, 20], Eaton and Lipsey (1978, 1979, 1980) [7, 8, 9], Salop (1979) [17], Gilbert and Harris (1984) [12], Bulow, Geanakoplos and Klemperer (1985) [3], Fudenberg and Tirole (1984) [11], and Sutton (1991) [21], among many others. In these studies, a range of firm commitment strategies were explored, including percentage of total sunk costs, outlay on the upkeep of sunk assets, research and development investment, advertising, and so on. However, in most of these studies, the setup features neither an infinite horizon framework nor a multiple-dimensional uncertainty assumption, whereas our model adopts the dynamic setting from Maskin and Tirole (1988) [14] instead of the static setting. Another feature that differentiates our work from the above literature is that we study price competition rather than quantity competition. Given that in reality firms tend to set their own prices in markets with differentiated products, our work shares the same focus on pricing strategy as Feng, Liu, Mazalov and Zheng (2019) [10], Lien, Mazalov, Melnik and Zheng (2016) [13], Mazalov and Melnik (2016) [15], and Mazalov, Chirkova, Zheng and Lien (2018) [16].

Our research is most related to Bloch, Eaton and Rothschild (2014), who provided a general framework to study the dynamic oligopoly problem with positioning investment [2]. The key difference between our setup and theirs is that we assume that the market demand is uncertain while there is no demand uncertainty in their framework. Another difference is that the number of firms is exogenously given in our work while they allow for the market equilibrium condition to determine the number of firms. It is also worthwhile to compare our work with Bonatti, Cisternas, and Toikka (2017). Given that we share a similar interest in studying dynamic oligopoly, we adopt a discrete-time framework with positioning investment requirement while they use a continuous-time approach with the assumption of asymmetric information on firms' production technology [1].

The rest of the chapter is organized as follows: In Section 2, we construct a dynamic model of Bertrand Competition and characterize the Symmetric Markov perfect Nash Equilibrium (SMPNE) of the game. In Section 3, we show the stage game market outcome, present the dynamic process of market structure, solve for the steady state of the dynamic system, and discuss about the speed of convergence to the steady state. Section 4 concludes the chapter and set directions for future work.

2 The Model

In a market that lasts for infinitely many periods, there are A firms which are able to produce differentiated goods with homogeneous production technologies. A firm has a position g, either established, E, or unestablished, U, so $g \in \{E,U\}$. Thus, in the market some of the participants are *established firms*, while others are *unestablished* in a given period. The number of *established firms* is $N \leq A$. A and N are integers, where A is exogenously given and N is endogenously determined through firms' decisions. The market demand d, can be either high, H, or low, L, in each period, so $d \in \{H,L\}$. For any given period, all established firms play a Bertrand game, while firms with no established position are not allowed to participate in the market competition. In every period, there is an opportunity for unestablished firms to make an investment in order to potentially establish their positions in the market for the next period, and there is also an opportunity for established firms to make an investment to maintain their positions in the market for the next period.

In each period, a representative firm will be in one of states defined with respect to its own position, the positions of the other firms and the condition of market demand. A firm's positioning investment, necessary to establish or maintain its position, determines whether it will have an established position or not in the next period, but has no effect on the oligopoly outcome within the current period. Positioning investments of all firms in the current period will determine their positions and the number of established firms (thus the market structure of oligopoly) in the next period. Thus, a Markov process for a representative firm can be defined in a space with $4A$ states. We focus on the symmetric Markov perfect Nash equilibrium (SMPNE) in which firms choose strategies to maximize the present value of their profit flow. In this work we focus on the market structure where firms compete by simultaneously setting their prices.

2.1 Positioning Investment

In any period, an unestablished firm can make an investment $I > 0$ to establish a position in the next period with probability P. If it does not choose to invest, it will remain an unestablished firm in the next period. If a firm already has an established position, it can invest $J > 0$ in order to main that position. Making such an investment results in a probability Q of maintaining its established position in the next period; however, failing to spend J leads to a loss of the established position in the next period with probability 1. We assume $0 < P < 1$ and $0 < Q < 1$. Notice that the positioning technology involves four parameters: I, J, P, and Q.

There are a number of explanations for the parameters I and J. I could be considered as product development, special purpose capital goods (including product-specific human capital) needed to produce the good, and/or a brand advertising campaign to launch the good. Similarly, J could be associated with product improvement, maintenance of product-specific capital goods, and/or maintenance of the good's brand.

There are two different market demand conditions in each period. The parameter H represents a high market demand and L represents a low market demand. In a given period, H occurs with probability h if the market demand condition was H in the previous period and occurs with probability $1-l$ if the market demand condition was L in the previous period. Similarly, L occurs with probability $1-h$ if the market demand condition was H in the previous period and occurs with probability l if the market demand condition was L in the previous period.

There are A firms in each market, each firm with an established or unestablished position, under an either high or low market demand condition. Therefore, by focusing on SMPNE, a representative firm can have in total $4A$ states in our dynamic model. In this work we consider three cases where $A \in \{1,2,3\}$. A firm in any period can be described as (g, n^E, d), where $g \in \{E,U\}$ means the representative firm's current position,

$n^E \in \{0,1,...,A-1\}$ is the number of the other firms which are currently established, and $d \in \{H,L\}$ is the demand condition of this market. For example, if the state of a representative firm is $(U,2,H)$, it means that the representative firm is currently unestablished, two of the other firms are currently established and the market demand condition is high. It is worth noting that the number of states for a firm is different from the number of states for the system, given our focus on SMPNE. the number of states for the system is $2(A+1)$. A state of the system can be presented as (m,d), where $m \in \{0,1,...,A\}$ is the number of established firms and $d \in \{H,L\}$ is the demand condition. For example, $(2,L)$ means two established firms with low market demand. For each of one-, two- and three-firm cases, we list all the states for a representative firm and all the states for the system as follows:

Table 1 States for a Representative Firm and States for the System ($A = 1$)

States for a Representative Firm	States for the System
(U, 0, H)	(0, H)
(U, 0, L)	(1, H)
(E, 0, H)	(0, L)
(E, 0, L)	(1, L)

Table 2 States for a Representative Firm and States for the System ($A = 2$)

States for a Representative Firm	States for the System
(U, 0, H)	(0, H)
(U, 1, H)	(1, H)
(U, 0, L)	(2, H)
(U, 1, L)	(0, L)
(E, 0, H)	(1, L)
(E, 1, H)	(2, L)
(E, 0, L)	
(E, 1, L)	

Table 3 States for a Representative Firm and States for the System ($A = 3$)

States for a Representative Firm	States for the System
(U, 0, H)	(0, H)
(U, 1, H)	(1, H)
(U, 2, H)	(2, H)
(U, 0, L)	(3, H)
(U, 1, L)	(0, L)
(U, 2, L)	(1, L)
(E, 0, H)	(2, L)
(E, 1, H)	(3, L)
(E, 2, H)	
(E, 0, L)	
(E, 1, L)	
(E, 2, L)	

Table 4 illustrates our notational convention for numbering the states for a representative firm. First, we enumerate the A states where $g = U$ and $d = H$, and the label for state (U, n^E, H) is $n^E + 1$. Next, we enumerate the A states where $g = U$ and $d = L$, and the label for state (U, n^E, L) is $A + n^E + 1$. Then, we enumerate the A states where $g = E$ and $d = H$, and the label for state (E, n^E, H) is $2A + n^E + 1$. Finally, we enumerate the A states where $g = E$ and $d = L$, and the label for state (E, n^E, L) is $3A + n^E + 1$.

Table 4 Labels for States for a Representative Firm

State	$(U,H,0;A)$	$\cdots$	$(U,H,A-1;A)$	$(U,L,0;A)$	$\cdots$	$(U,L,A-1;A)$	$(E,H,0;A)$	$\cdots$	$(E,H,A-1;A)$	$(E,L,0;A)$	$\cdots$	$(E,L,A-1;A)$
Number	1	$\cdots$	A	$A+1$	$\cdots$	$2A$	$2A+1$	$\cdots$	$3A$	$3A+1$	$\cdots$	$4A$

A firm's strategy of the game consists of two components. The first component is the positioning investment strategy for every period, and the second component is the pricing strategy for every period in which the firm is established. Since we focus on SMPNE, any decision node of the firm will be fully described by one of the $4A$ states. To put it in another way, the positioning and pricing strategies of a firm at any decision node depend only on the firm's state at that decision node. In the dynamic game, firms' positioning and pricing strategies have different roles. In a given period, the positioning strategies of firms jointly determine a Markov process that predicts their states in the next period, while the pricing strategies of established firms jointly determine their profits in that period. Therefore, we use a two-step procedure to formulate the value function under SMPNE: We first solve for the equilibrium of the static stage Bertrand game, and then use the associated stage game equilibrium payoff to formulate the dynamic game.

2.2 The Static Stage Bertrand Game

Suppose that the established firms produce symmetrically differentiated goods. Following the well accepted quadratic utility setup in the literature (Dixit, 1979; Bloch, Eaton and Rothschild, 2014; among many others), the representative consumer's utility function is specified as below:

$$U(y, q_1, q_2, ..., q_N) = y + \alpha_d \sum_{i=1,...,N} q_i - \frac{\beta}{2} \sum_{i=1,...,N} q_i^2 - \gamma \sum_{i=1,...,N, j\neq i} q_i q_j \quad d = H \text{ or } L, \tag{1}$$

where y is the expenditure on a composite good and q_i is the quantity of the goods produced by the i^{th} established firm. We require $\beta > \gamma > 0$ and $\alpha_H > \alpha_L > 0$. Such a utility function has several nice features: First, it is quasilinear with respect to the composite good (the numeraire); Second, Since $\beta > 0$, it represents a diminishing marginal utility for good i, where $i = 1, ..., N$; Third, Since $\gamma > 0$, any two goods i, j with$i \neq j$ are substitutes, where $i, j = 1, ..., N$; Fourth, as will be shown next, it implies a linear demand function.

Maximizing the representative consumer's utility gives the inverse demand functions for the N differentiated goods:

$$p_i = \alpha_d - \beta q_i - \gamma \sum_{j\neq i} q_j, i = 1, ..., N, j = 1, ..., N, d = H \text{ or } L. \tag{2}$$

We assume that firms simultaneously choose their own prices to maximize their profits in a Bertrand competition setup. We also assume that all firms have the same constant marginal production cost, which is set to zero without loss of generality.

Using the inverse demand functions in equation (2), we obtain the following demand functions:

$$q_i = A_N - B_N p_i + C_N \sum_{j \neq i} p_j, i = 1, ..., N, \tag{3}$$

where A_N, B_N and C_N are parameters defined in the following table:

N	A_N	B_N	C_N
1	$\frac{\alpha_d}{\beta}$	$\frac{1}{\beta}$	
2	$\frac{\alpha_d}{\beta+\gamma}$	$\frac{\beta}{\beta^2-\gamma^2}$	$\frac{\gamma}{\beta^2-\gamma^2}$
3	$\frac{\alpha_d}{\beta+2\gamma}$	$\frac{\beta+\gamma}{\beta^2+\beta\gamma-2\gamma^2}$	$\frac{\gamma}{\beta^2+\beta\gamma-2\gamma^2}$

The profit of a representative firm (assuming firm 1 without loss of generality), is

$$\pi_1 = p_1(A_N - B_N p_1 + C_N \sum_{j \neq 1} p_j) \tag{4}$$

The first order condition for profit-maximization is

$$A_N - 2B_N p_1 + C_N \sum_{j \neq 1} p_j = 0 \tag{5}$$

By equation (5), we can easily derive the equilibrium price, denoted by $p_B^*(N,d)$. We can then solve for the equilibrium quantity, denoted by $q_B^*(N,d)$, the oligopoly profit of a representative firm in equilibrium, denoted by $R_B^*(N,d)$, the equilibrium consumers' surplus, denoted by $CS_B^*(N,d)$, and the equilibrium total surplus, denoted by $TS_B^*(N,d)$. They are shown in the following table.

N	$p_B^*(N,d)$	$q_B^*(N,d)$	$R_B^*(N,d)$	$CS_B^*(N,d)$	$TS_B^*(N,d)$
1	$\frac{\alpha_d}{2}$	$\frac{\alpha_d}{2\beta}$	$\frac{\alpha_d^2}{4\beta}$	$\frac{\alpha_d^2}{8\beta}$	$\frac{3\alpha_d^2}{8\beta}$
2	$\frac{\alpha_d(\beta-\gamma)}{2\beta-\gamma}$	$\frac{\alpha_d\beta}{(\beta+\gamma)(2\beta-\gamma)}$	$\frac{\alpha_d^2\beta(\beta-\gamma)}{(\beta+\gamma)(2\beta-\gamma)^2}$	$\frac{\alpha_d^2\beta^2}{(\beta+\gamma)(2\beta-\gamma)^2}$	$\frac{\alpha_d^2\beta(3\beta-2\gamma)}{(\beta+\gamma)(2\beta-\gamma)^2}$
3	$\frac{\alpha_d(\beta-\gamma)}{2\beta}$	$\frac{\alpha_d(\beta+\gamma)}{(\beta+2\gamma)(2\beta)}$	$\frac{\alpha_d^2(\beta+\gamma)(\beta-\gamma)}{(\beta+2\gamma)(2\beta)^2}$	$\frac{3\alpha_d^2(\beta+\gamma)^2}{2(\beta+2\gamma)(2\beta)^2}$	$\frac{3\alpha_d^2(\beta+\gamma)(3\beta-\gamma)}{2(\beta+2\gamma)(2\beta)^2}$

Note that the equilibrium price, the equilibrium quantity, the stage game equilibrium profit per firm, the stage game equilibrium consumers' surplus, and the stage game equilibrium total surplus, are all dependent on the number of established firms, N, and the condition of market demand, d.

Recall from Table 4 that if the labeling number for the state of a firm is $k \in \{2A+1, ..., 3A\}$, then $g = E$ (the firm is established), $d = H$ and $n^E = k - 2A - 1$. Consequently, the equilibrium price in state $k \in \{2A+1, ..., 3A\}$ is $p_B^*(k-2A, H)$. If the labeling number for the state of a firm is $k \in \{3A+1, ..., 4A\}$, then $g = E$ (the firm is established), $d = L$ and $n^E = k - 3A - 1$, and the equilibrium price in state $k \in \{3A+1, ..., 4A\}$ is $p_B^*(k-3A, L)$. Similarly, the profit and total surplus are $R_B^*(k-2A, H)$ and $TS_B^*(k-2A, H)$, respectively, in any state $k \in \{2A+1, ..., 3A\}$, and they are $R_B^*(k-3A, L)$ and $TS_B^*(k-3A, L)$, respectively, in any state $k \in \{3A+1, ..., 4A\}$.

2.3 Value Function and Transition Matrix

Given a common strategy for all the other firms, it suffices to focus on the payoff-maximizing decisions of a representative firm. The probability of making a positioning investment by the representative firm in state k is denoted by s_R^k, $0 \leq s_R^k \leq 1$. The relevant investment is I when $k \in \{1,...,2A\}$ since its position is U, and it is J when $k \in \{2A+1,...,4A\}$ since its position is E. The representative firm's positioning strategy is then $S_R = (s_R^1, s_R^2, ..., s_R^{4A})$ and the positioning strategy of the other firms is $S_O = (s_O^1, s_O^2, ..., s_O^{4A})$. Given firms' positioning strategies, we can define the transition matrix $T(s_R, s_O)$ for a representative firm, a $4A$ by $4A$ matrix that specifies a representative firm's probability of transition between any two states. For example, an entry in the transition matrix, say $T_{kl}(s_R, s_O)$, denotes the representative firm's probability of transition from state k in any period to state l in the next period. Note that the transition matrix not only depends on the positioning investment strategy of the representative firm and other firms but also depends on the probabilities for high or low market demand conditions.

To illustrate the concept of transition matrix more explicitly, we consider the case where $A = 3$ and use $T_{24}(s_R, s_O)$, an entry in the transition matrix indicating the representative firm's probability of transition from state 2 to state 4, as an example. First notice that when the representative firm is in state 2 $(U,1,H)$, one of the other two firms is also in state 2 $(U,1,H)$, and the other one is in state 7 $(E,0,H)$. The representative firm will be in state 4 $(U,0,L)$ in the next period if the following four independent events all occur: the representative firm's position remains U, the position of the other firm that is currently in state 2 remains U, the position of the other firm that is currently in state 7 switches to U and the market demand shifts from high to low. The first of these events will occur with probability $1 - s_R^2 P$, the second with probability $1 - s_O^2 P$, the third with probability $1 - s_O^7 Q$, and the last with probability $1 - h$ so

$$T_{24}(S_R, S_O) = (1 - s_R^2 P)(1 - s_O^2 P)(1 - s_O^7 Q)(1 - h) \tag{6}$$

In states 1 through $2A$, the profit of the firm is zero because the representative firm has an unestablished position in these states. Following the notational convention in Table 4, we denote the representative firm's operating profit when it is in state $k \in \{2A+1,...,3A\}$ as $R_B(k-2A, H)$, and $R_B(k-2A, H) > 0$ because the representative firm has an established position and plays the Bertrand game in states $2A+1$ to $3A$. Similarly we also have $R_B(k-3A, L) > 0$ for states $3A+1$ to $4A$. To obtain the representative firm's cash flow in state k, denoted by π_B^k, we subtract from $R_B(k-2A, d)$ (for $k \in \{2A+1,...,3A\}$) or $R_B(k-3A, d)$ (for $k \in \{3A+1,...,4A\}$) the expected costs associated with its positioning investment.

$$\pi_B^k = 0 - s_R^k I \text{ if } k \in \{1,...,2A\}, \tag{7}$$
$$\pi_B^k = R_B(k-2A, H) - s_R^k J \text{ if } k \in \{2A+1,...,3A\}, \tag{8}$$
$$\pi_B^k = R_B(k-3A, L) - s_R^k J \text{ if } k \in \{3A+1,...,4A\} \tag{9}$$

Let $V^k((\widehat{S}_R, S_R), S_O)$ denote the present value of the representative firm's profit over an infinite time horizon, when it is in state k. Here, $\widehat{S}_R = (\widehat{s}_R^1, \widehat{s}_R^2, ..., \widehat{s}_R^{4A})$ means the representative firm's positioning investment strategy in the current period, S_R means the representative firm's positioning investment strategy in all subsequent periods, and S_O means other firms' positioning investment strategy in current and subsequent periods. The value function can be calculated in the following expression:

$$V^k((\widehat{S}_R, S_R), S_O) = \pi_B^k + D \sum_{j=1,...,4A} T_{kj} V^j((\widehat{S}_R, S_R), S_O), k = 1,...,4A \tag{10}$$

where $D, 0 < D < 1$, is the discount factor.

2.4 Characterizing the Symmetric Markov Perfect Nash Equilibrium

In order for S^* to be the symmetric Markov perfect Nash equilibrium strategy, two conditions should hold:

For all $k \in \{1, ..., 4A\}$,

$$V^k((S^*, S^*), S^*) \geq V^k(((s^k, S^*_{-k}), S^*), S^*), \forall s^k \in [0,1]. \tag{11}$$

If there exists some k such that $0 < s^{k^*} < 1$, then

$$V^k((S^*, S^*), S^*) = V^k(((s^k, S^*_{-k}), S^*), S^*), \forall s^k \in [0,1]. \tag{12}$$

Note that S^*_{-k} denotes the representative firm's equilibrium strategy profile in the current period for all states other than k, s^{k^*} denotes the representative firm's equilibrium strategy profile in the current period for state k, and s^k denotes the representative firm's any possible strategy profile in the current period for state k. $0 < s^{k^*} < 1$ implies that the firm is indifferent between making the positioning investment and not making the investment for state k.

By Conditions (11) and (12), we are able to solve for the symmetric Markov perfect Nash Equilibrium (SMPNE) strategy, $S^* = (s^{1^*}, s^{2^*}, ..., s^{4A^*})$, by using the algorithm described in Appendix.

3 Results

In the model, there are 11 exogenous variables: six parameters that control demand of the representative consumer (α_H, α_L, β, γ, h, l), four parameters that govern positioning technology (I, J, P, Q), and a discount factor (D). In a given parameter environment, firms conduct Bertrand competition in every stage. As a baseline environment, we set the demand parameters to be $\alpha_H = 60$ for the high demand market and $\alpha_L = \frac{60}{\sqrt{2}}$ for the low demand market. Besides, we assume $\beta = 1$, $\gamma = 0.95$, $h = 0.7$, and $l = 0.7$. In addition, the positioning technology parameters are $I = 700$, $J = 200$, $P = 0.8$ and $Q = 0.95$.

In the following, we first present the stage games results under the baseline setup, then provide evidence for market dynamic process, and later we study the steady state of the dynamics as well as the speed of convergence to the steady state.

3.1 The Stage Game Outcome under Baseline Parameterization

Table 5 shows the equilibrium firm profit, consumers' surplus, and total surplus in the Bertrand competition under different market demand condition (H or L), with different numbers of established firms (one, two, or three), respectively. It is easy to see that the market demand condition affects firms' profit, consumers' surplus and total surplus in the same direction. We also observe that while firms' profit decrease with the

Table 5 Stage Payoffs under Different States in the Bertrand Framework

Demand Condition	Number of Firms	Payoffs	
High	One	Profit Per Firm	900
		Consumers' Surplus	450
		Total Surplus	1350
	Two	Profit Per Firm	84
		Consumers' Surplus	1674
		Total Surplus	1842
	Three	Profit Per Firm	30
		Consumers' Surplus	1770
		Total Surplus	1860
Low	One	Profit Per Firm	450
		Consumers' Surplus	225
		Total Surplus	675
	Two	Profit Per Firm	42
		Consumers' Surplus	837
		Total Surplus	921
	Three	Profit Per Firm	15
		Consumers' Surplus	885
		Total Surplus	930

number of firms, both consumers' surplus and total surplus increase with the number of firms. This implies that although the gain from the consumers' side is partially balanced by the loss from the producers' side, overall the society benefits from having a more competitive environment.

3.2 Realization of the Dynamic Process

From Tables 6 to 11, we provide an example of the two-firm case with a realization of a 20-period dynamic process under the Bertrand competition by using the baseline parameter values. Each table represents a realized dynamic process for one of the 6 initial states, as a combination of market demand condition and number of established firms. As shown in these tables, the dynamic patterns of realization differ significantly with different initial states for the system. In each period of time, the number of established firms can be either the same or different from the previous period. A similar pattern holds for conditions of market demand as well. From the patterns of realization, it is not easy to get a firm grip on what is happening. Because of this difficulty, we later focus on the steady-state properties of the model.

Table 6 20-Period Dynamic Process For Two Firms with the Initial State for the System $(0,H)$

Number of Established Firms	Period of Time																			
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	X															X				
1		X	X	X	X	X	X				X	X	X	X	X					X
2								X	X	X							X	X	X	
Market Demand																				
H	X	X	X	X	X									X	X	X	X	X	X	X
L						X	X	X	X	X	X	X	X							
Profit Per Firm	0	900	900	900	900	450	450	42	42	42	450	450	450	900	900	0	84	84	84	900
Consumers' Surplus	0	450	450	450	450	225	225	837	837	837	225	225	225	450	450	0	1647	1674	1674	450
Total Surplus	0	1350	1350	1350	1350	675	675	921	921	921	675	675	675	1350	1350	0	1842	1842	1842	1350

Table 7 20-Period Dynamic Process For Two Firms with the Initial State for the System $(1,H)$

Number of Established Firms	Period of Time																			
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0									X	X						X				
1	X	X	X	X	X	X	X	X					X	X	X					X
2											X	X					X	X	X	
Market Demand																				
H	X	X	X	X	X	X	X	X								X	X	X	X	X
L									X	X	X	X	X	X	X					
Profit Per Firm	900	900	900	900	900	900	900	900	0	0	42	42	450	450	450	0	84	84	84	900
Consumers' Surplus	450	450	450	450	450	450	450	450	0	0	837	837	225	225	225	0	1674	1674	1674	450
Total Surplus	1350	1350	1350	1350	1350	1350	1350	1350	0	0	921	921	675	675	675	0	1842	1842	1842	1350

Table 8 20-Period Dynamic Process For Two Firms with the Initial State for the System $(2,H)$

Number of Established Firms	Period of Time																			
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0											X	X								
1				X	X	X	X	X	X	X			X	X	X	X				
2	X	X	X														X	X	X	X
Market Demand																				
H	X	X	X	X	X	X	X										X	X	X	X
L								X	X	X	X	X	X	X	X	X				
Profit Per Firm	84	84	84	900	900	900	900	450	450	450	0	0	450	450	450	450	84	84	84	84
Consumers' Surplus	1674	1674	1674	450	450	450	450	225	225	225	0	0	225	225	225	225	1674	1674	1674	1674
Total Surplus	1842	1842	1842	1350	1350	1350	1350	675	675	675	0	0	675	675	675	675	1842	1842	1842	1842

Table 9 20-Period Dynamic Process For Two Firms with the Initial State for the System $(0, L)$

Number of Established Firms	Period of Time																			
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	X															X				
1		X	X	X	X	X	X				X	X	X	X	X					X
2								X	X	X							X	X	X	
Market Demand																				
H						X	X	X	X	X	X	X	X							
L	X	X	X	X	X									X	X	X	X	X	X	X
Profit Per Firm	0	450	450	450	450	900	900	84	84	84	900	900	900	450	450	0	42	42	42	450
Consumers' Surplus	0	225	225	225	225	450	450	1674	1674	1674	450	450	450	225	225	0	837	837	837	225
Total Surplus	0	675	675	675	675	1350	1350	1842	1842	1842	1350	1350	1350	675	675	0	921	921	921	675

Table 10 20-Period Dynamic Process For Two Firms with the Initial State for the System $(1, L)$

Number of Established Firms	Period of Time																			
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0									X	X						X				
1	X	X	X	X	X	X	X	X					X	X	X					X
2											X	X					X	X	X	
Market Demand																				
H									X	X	X	X	X	X	X					
L	X	X	X	X	X	X	X	X								X	X	X	X	X
Profit Per Firm	450	450	450	450	450	450	450	450	0	0	84	84	900	900	900	0	42	42	42	450
Consumers' Surplus	225	225	225	225	225	225	225	225	0	0	1674	1674	450	450	450	0	837	837	837	225
Total Surplus	675	675	675	675	675	675	1350	675	0	0	1842	1842	1350	1350	1350	0	921	921	921	675

Table 11 20-Period Dynamic Process For Two Firms with the Initial State for the System $(2, L)$

Number of Established Firms	Period of Time																			
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0											X	X								
1				X	X	X	X	X	X	X			X	X	X	X				
2	X	X	X														X	X	X	X
Market Demand																				
H								X	X	X	X	X	X	X	X	X				
L	X	X	X	X	X	X	X										X	X	X	X
Profit Per Firm	42	42	42	450	450	450	450	900	900	900	0	0	900	900	900	900	42	42	42	42
Consumers' Surplus	837	837	837	225	225	225	225	450	450	450	0	0	450	450	450	450	837	837	837	837
Total Surplus	921	921	921	675	675	675	675	1350	1350	1350	0	0	1350	1350	1350	1350	921	921	921	921

3.3 Conditional Probabilities

Instead of further investigating the realized outcome of the dynamic process, we now study the probabilistic distribution of states in the dynamic process. From the tables in Subsection 3.2, it should be clear that the system can go from a particular state to a number of different states, including the state that it is currently in, within a given number of periods. This raises a natural question: What is the probability of going from one state to another in 1 period, in 2 periods, and so forth? We will give an answer to this question below, assuming all firms follow the equilibrium behavior. Thus, the probability for a particular state in period $t > 0$, conditional on being in a particular state in period $t = 0$, is calculated when firms use their SMPNE strategies. The firm's expected probability distribution over states in period t can be derived as follows:

$$E_t = E_0[T^*]^t, \tag{13}$$

where E_0 is the initial probability distribution over states which assigns value 1 to the initial state and 0 to all other states, and T^* is the equilibrium transition matrix for the system.

We use the two-firm case ($A = 2$)) as an illustrative example. The number of states for the system is thus 6. We present the conditional probability distribution of going from any of the 6 states to the same state in one period. In Table 12, we observe that the probability to stay in the initial state of the system at time $t = 1$ conditional on being in this particular state at time $t = 0$ is very low if the initial state for the system is $(0,H)$ or $(0,L)$, and such a probability is higher than 0.5 for all other initial states for the system. Recall that the state for the system depends on two factors: the number of established firms and the condition of the market demand. The result in Table 12 shows that staying in the same state is unlikely when there is no established firm in the market, regardless of the market demand condition.

Table 12 Conditional Probabilities of Remaining in the Same State Within 1 Period ($A = 2$)

States for the System	Conditional Probability
(0, H)	0.050
(1, H)	0.665
(2, H)	0.632
(0, L)	0.062
(1, L)	0.665
(2, L)	0.622

As shown in Table 13, we also present the transition probabilities with all possible initial states for the system in period 0. It is easy to see that the transition probabilities update significantly during the early periods and the size of changes becomes smaller and smaller over time. Furthermore, the differences across the transition probabilities with different initial states seem to decrease over time as well. These observations imply that the transition probability distribution will converge to a steady state. In other words, the probability distribution of the states for the system after sufficiently large number of periods will be the same regardless the initial state for the system.

Table 13 Transition Probabilities with Different Initial Sates for the System ($A = 2$)

State for the System	Period 0	Period 1	Period 2	Period 3	Period 4	Period 5	Period 6	Period 7	Period 8	Period 9	Period 10
(0, H)	1	0.050	0.015	0.014	0.014	0.015	0.016	0.016	0.017	0.017	0.018
(1, H)	0	0.275	0.263	0.262	0.270	0.281	0.293	0.303	0.313	0.321	0.328
(2, H)	0	0.375	0.302	0.256	0.228	0.209	0.194	0.181	0.171	0.162	0.154
(0, L)	0	0.022	0.011	0.012	0.014	0.015	0.016	0.017	0.017	0.018	0.018
(1, L)	0	0.118	0.192	0.233	0.259	0.278	0.292	0.304	0.314	0.323	0.330
(2, L)	0	0.161	0.217	0.223	0.215	0.202	0.190	0.179	0.169	0.160	0.152
(0, H)	0	0.035	0.030	0.027	0.025	0.024	0.024	0.023	0.023	0.023	0.022
(1, H)	1	0.665	0.535	0.478	0.451	0.436	0.426	0.418	0.412	0.407	0.403
(2, H)	0	0.000	0.015	0.027	0.037	0.045	0.053	0.059	0.065	0.070	0.075
(0, L)	0	0.015	0.022	0.024	0.024	0.024	0.024	0.023	0.023	0.023	0.022
(1, L)	0	0.285	0.388	0.421	0.429	0.427	0.423	0.418	0.413	0.408	0.404
(2, L)	0	0.000	0.011	0.023	0.034	0.044	0.051	0.058	0.064	0.069	0.074
(0, H)	0	0.002	0.004	0.006	0.008	0.010	0.011	0.012	0.013	0.014	0.015
(1, H)	0	0.067	0.104	0.135	0.164	0.190	0.214	0.235	0.254	0.270	0.284
(2, H)	1	0.632	0.472	0.391	0.341	0.305	0.277	0.253	0.233	0.216	0.201
(0, L)	0	0.001	0.003	0.006	0.008	0.010	0.011	0.013	0.014	0.015	0.015
(1, L)	0	0.029	0.077	0.121	0.158	0.189	0.215	0.237	0.256	0.272	0.286
(2, L)	0	0.271	0.340	0.341	0.321	0.296	0.272	0.250	0.230	0.213	0.198
(0, H)	0	0.026	0.012	0.013	0.014	0.015	0.016	0.017	0.017	0.018	0.018
(1, H)	0	0.125	0.203	0.243	0.268	0.286	0.299	0.310	0.319	0.327	0.333
(2, H)	0	0.148	0.205	0.212	0.204	0.194	0.182	0.172	0.163	0.155	0.148
(0, L)	1	0.062	0.017	0.015	0.015	0.016	0.016	0.017	0.018	0.018	0.018
(1, L)	0	0.292	0.282	0.279	0.285	0.294	0.304	0.313	0.321	0.329	0.335
(2, L)	0	0.346	0.281	0.238	0.213	0.195	0.182	0.171	0.161	0.153	0.147
(0, H)	0	0.015	0.022	0.024	0.024	0.024	0.023	0.023	0.023	0.023	0.022
(1, H)	0	0.285	0.388	0.421	0.429	0.427	0.423	0.417	0.412	0.407	0.403
(2, H)	0	0.000	0.011	0.023	0.034	0.044	0.052	0.059	0.065	0.070	0.074
(0, L)	0	0.035	0.030	0.027	0.025	0.024	0.024	0.023	0.023	0.023	0.023
(1, L)	1	0.665	0.535	0.479	0.452	0.437	0.427	0.419	0.413	0.408	0.404
(2, L)	0	0.000	0.015	0.026	0.035	0.044	0.052	0.058	0.064	0.069	0.073
(0, H)	0	0.001	0.003	0.006	0.008	0.010	0.011	0.013	0.014	0.015	0.015
(1, H)	0	0.032	0.082	0.126	0.162	0.193	0.218	0.239	0.258	0.274	0.287
(2, H)	0	0.267	0.335	0.336	0.317	0.292	0.269	0.247	0.228	0.212	0.197
(0, L)	0	0.002	0.005	0.007	0.009	0.010	0.012	0.013	0.014	0.015	0.016
(1, L)	0	0.076	0.115	0.146	0.174	0.199	0.222	0.243	0.260	0.276	0.289
(2, L)	1	0.622	0.460	0.379	0.330	0.296	0.268	0.245	0.226	0.209	0.195

3.4 The Steady State of the System

The steady state can be obtained when the power of the equilibrium transition matrix, t, approaches infinity such that the probability distribution becomes constant, by Equation (13).

First, we obtain the steady state of the system by transferring from the steady state of a representative firm. Note that the steady state of a representative firm depends on the positional condition of the representative firm, the number of other established firms, and the condition of the market demand. However, the state for the system only focuses on the number of established firms and the condition of the demand. The steady state of the system is presented by a probability distribution of the states for the system in the far future. This distribution is used to predict the current period if we have no information about the history of the dynamic model. In Table 14, we observe that $(1,H)$ and $(1,L)$ are most likely to occur in the steady state, while $(0,H)$

and $(0, L)$ are almost unlikely. Having two established firms competing in the market is possible in the steady state but the chances are very low.

Table 14 Steady State of the System ($A = 2$)

States for the System	Probability
(0, H)	0.021
(1, H)	0.376
(2, H)	0.103
(0, L)	0.021
(1, L)	0.377
(2, L)	0.102

Similarly, we provide the steady state probability distribution for the one-firm case and for the three-firm case, respectively. It can be easily seen by comparing Tables 14, 15, and 16, having only one established firm in the market is most likely in the steady state under the Bertrand competition framework.

Table 15 Steady State of the System ($A = 1$)

States for the System	Probability
(0, H)	0.029
(1, H)	0.471
(0, L)	0.029
(1, L)	0.471

Table 16 Steady State of the System ($A = 3$)

States for the System	Probability
(0, H)	0.023
(1, H)	0.380
(2, H)	0.093
(3, H)	0.004
(0, L)	0.023
(1, L)	0.381
(2, L)	0.092
(3, L)	0.004

3.5 Expected Duration

Using the information on the conditional probabilities of remaining in the same state for 1 period, obtained from Table 12, we can derive the expected duration of each state for the system. The formula to calculate the expected duration is as follows:

$$ED_k = \frac{1}{1-CP_k}, k \in \{1,...,2A+2\}. \quad (14)$$

In the above formula, CP_k is the conditional probability of remaining in state k for the system for 1 period and ED_k is the expected duration for state k for the system. Again, we use the two-firm case as an example. In Table 17, the first column shows the initial states for the system. For each of these initial states, the conditional probability to stay in that state for 1 period is same as the one in Table 12. We use formula 14 to calculate the expected duration for each state, as shown in the third column. Consistent with our observation from Table 12, since staying in the same state is unlikely when there is no established firm in the market, the expected duration for such a state is indeed about 1 period. Since the chance for more than one established firm to exist in the market is more than 0.5 for every such state, the expected duration for each such state is indeed more than 2 periods.

Table 17 Conditional Probabilities and the Expected Duration ($A = 2$)

States for the System	Conditional Probability	Expected Duration
(0, H)	0.050	1.053
(1, H)	0.665	2.985
(2, H)	0.632	2.716
(0, L)	0.062	1.066
(1, L)	0.665	2.985
(2, L)	0.622	2.646

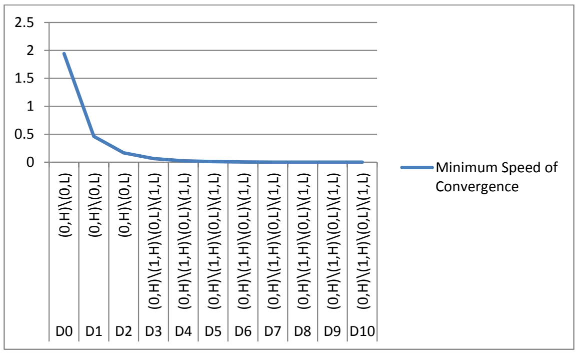
3.6 The Speed of Convergence to the Steady State

Given the existence of the steady state, it is of our interest to further explore how fast the equilibrium dynamic process converges to the steady state. In Figure 1, we show the speed of convergence for three cases: $A = 1,2,3$, respectively. For Panels (a), (c), (e) in Figure 1, the horizontal axis indicates for any given length of periods t (denoted by Dt, where $t = 0,...,10$) which initial state has the minimum converging speed to the steady state (specified in Tables 15, 14, 16, respectively). For Panels (b), (d), (f) in Figure 1, the horizontal axis instead indicates the initial state that has the maximum converging speed. The vertical axis in all 6 panels represents the magnitude of the minimum (or maximum) converging speed among all the initial states, for any given length of periods t. Note that the converging speed of a given initial state for a t length of periods, is measured by the sum of absolute values of the differences between the period-t probability and the steady-state probability for each state, where the period-t probability is calculated by using Formula (13). It is worth mentioning that a lower converging speed has a greater measure and a higher converging speed has a smaller measure by our definition.

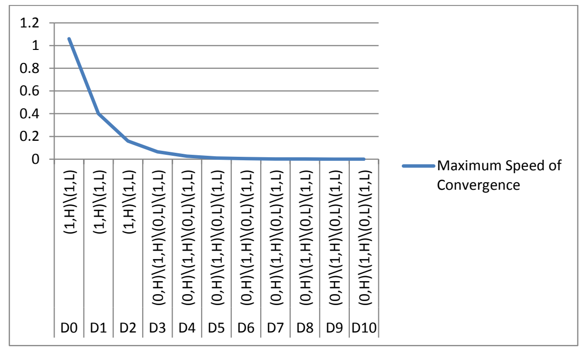
Panels (a) and (b) in Figure 1 show the minimum speed and the maximum speed of convergence for the one-firm case. We observe that the initial states $(0,H)$ and $(0,L)$ have minimum speed of convergence for any length of periods no more than 3, and after that all the initial states have the same speed of convergence.

In contrast, the initial states $(1,H)$ and $(1,L)$ have the maximum speed of convergence for any length of periods no more than 3.

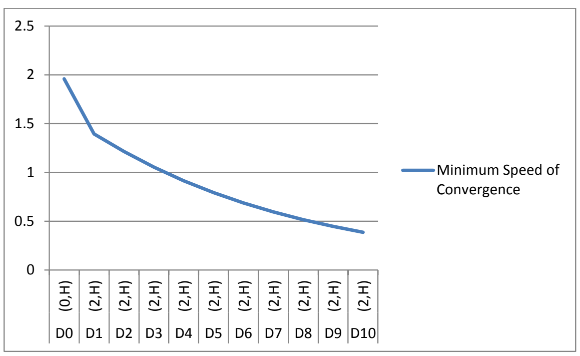
Fig. 1 The Speed of Convergence in the Bertrand Framework



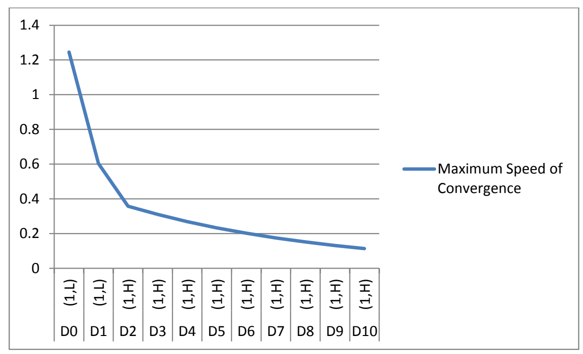
(a) Minimum Speed of Convergence (One Firm)



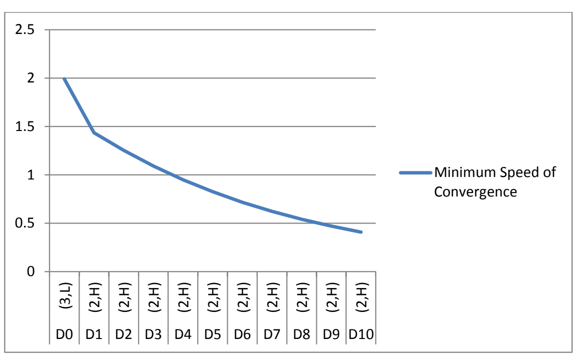
(b) Maximum Speed of Convergence (One Firm)



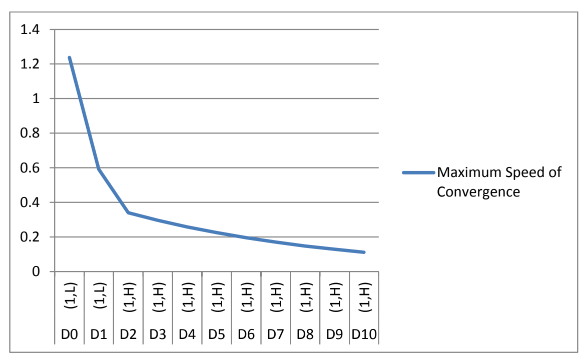
(c) Minimum Speed of Convergence (Two Firms)



(d) Maximum Speed of Convergence (Two Firms)



(e) Minimum Speed of Convergence (Three Firms)



(f) Maximum Speed of Convergence (Three Firms)

The results for the two-firm case are slightly different. Panel (c) in Figure 1 indicates that the initial state $(2,H)$ has the minimum speed of convergence and Panel (d) in Figure 1 indicates that the initial state $(1,H)$ has the maximum speed of convergence for any length of periods no less than 2.

For the three-firm case the results about the converging speed are almost the same as those for the two-firm case. It is still the initial state $(2,H)$ that has the minimum speed of convergence (Panel (e) in Figure 1) and it is still the initial state $(1,H)$ that has the maximum speed of convergence (Panel (f) in Figure 1) for any length of periods no less than 2. The only difference between these two cases is that in the two-firm case state $(0,H)$ has the largest distance from the steady state while in the three-firm case it is the state $(3,L)$ that has the largest distance from the steady state.

If we set the convergence criteria to be such that the distance between the probability distribution for the period-t state and the probability distribution for the steady state should be less than 0.01, then obviously there is no convergence for the first ten periods for the two- or three-firm case. However, if we allow for t to be sufficiently large, we can guarantee that the probability distribution for the period-t will converge to the probability distribution for the steady state, for both the two-firm case and the three-firm cases.

4 Conclusion

In this work we study the dynamic interaction among a fixed number of firms in a Bertrand market with stochastic entry and stochastic demand. A firm's entry into market in the next period is possible by making a positioning investment, while both the cost and the success rate of such an investment depend on the firm's position (either established or unestablished) in the current period. After all established firms successfully enter the market for the current period, the market demand is realized, and all established firms play a Bertrand game by simultaneously setting their own prices for their differentiated goods. Unestablished firms have no opportunity to enter the market for the current period.

We characterize the symmetric Markov perfect Nash Equilibrium (SMPNE) of such a dynamic game, where a firm's strategy consists of two components: strategy for positioning investment which determines the firm's position in the next period, and strategy for setting the price which affects the firm's profit in the current period. By considering three different cases where the number of firms in the market is 1, 2, and 3, respectively, we show the stage game market outcome, present the dynamic process of market structure, solve for the steady state of the dynamic system, and discuss about the speed of convergence to the steady state.

Our work contributes to the dynamic oligopoly literature by allowing for two dimensions of stochastic uncertainty in firms' decision-making. Firms have to take into account both the uncertainty of the result from the positioning investment and the uncertainty of the market demand condition given successful investment, when making the positioning investment decision.

The current analysis focuses on Bertrand competition and a potential direction for future work is to study firms' dynamic interaction under other market structures. Since the number of firms is assumed exogenously fixed in the current framework, one may consider an endogenously determined market structure with two dimensions of uncertainty. Other possibilities to extend the current work include considering the externality of positioning investment and/or allowing for the interdependence between market condition and positioning investment.

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Appendix

Algorithm for Searching for the Symmetric Markov Perfect Nash Equilibrium

In order to solve for the symmetric Markov perfect Nash equilibrium strategy, we use the following three-step algorithm.

Step 1. We set the parameter value of exogenous variables α_H, α_L, β, γ, I, J, P, Q, h, l, D, and calculate the stage game equilibrium firm profit $R(N,d)$. We provide the initial value of Markov strategy $S=(s^1,s^2,...,s^{4A})$.

Step 2. We set $\widehat{S}_R=S, S_R=S$ and $S_O=S$ in the system of $4A$ value functions specified by equations (10) and obtain $V^k((S,S),S), k=1,...,4A$. We use these values to calculate s^k and ns^k by implementing the following three criteria:

(1) If $V^k(((1,S_{-k}),S),S)>V^k(((0,S_{-k}),S),S)$,
then $ns^k=\min(1,s^k+\varepsilon(V^k(((1,S_{-k}),S),S)-V^k(((0,S_{-k}),S),S)))$.
(2) If $V^k(((1,S_{-k}),S),S)<V^k(((0,S_{-k}),S),S)$,
then $ns^k=\max(0,s^k-\varepsilon(V^k(((0,S_{-k}),S),S)-V^k(((1,S_{-k}),S),S)))$.
(3) If $V^k(((1,S_{-k}),S),S)=V^k(((0,S_{-k}),S),S)$,
then $ns^k=s^k$. We set $\varepsilon=.000005$ for this exercise.

Step 3. We check the condition of convergence as below:

$$\Delta=\sum_{1}^{4A}\left|s^k-ns^k\right|\leq\delta.$$

We set $\delta=.0000000001$ in practice. The algorithm will deliver the equilibrium strategy only if $\Delta\leq\delta$. If $\Delta>\delta$. we set $S=(s^1,s^2,...,s^{4A})=(ns^1,ns^2,...,ns^{4A})$, and go back to Step 2.

The algorithm runs until $\Delta\leq\delta$.

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