Multi-period Matching with Commitment *

Mu Zhang[†] Jie Zheng[‡]

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Abstract

Many multi-period matching markets exhibit some level of commitment. That is, agents' ability to terminate an existing relationship may be restricted by cost of breakups, binding contracts or social norms. This paper models matching markets with three types of commitment, defines corresponding notions of stability and examines the existence of stable mechanisms, as well as specifies sufficient conditions for efficiency, strategy-proofness and other properties. Firstly, the market with full commitment most closely resembles the static matching market, where most of the results, such as existence of stability, hold in the most general class of preferences. However, there is no dynamically stable spot rule, which only depends on spot markets, unless agents are extremely impatient. Secondly, for the models with two-sided commitment or one-sided commitment, desirable properties that are valid under the setup with a fixed set of individuals may not hold when arrivals and departures are introduced, and three approaches are proposed to deal with this issue. Whenever a dynamically stable matching exists, we construct an algorithm building upon the Deferred Acceptance algorithm of Gale and Shapley (1962) to charactrize such a matching outcome. Moreover, as extensions, we discuss the case with no commitment and conduct welfare comparisons among cases with different types of commitment.

Keywords: Multi-period Matching, Commitment, Dynamic Stability, Deferred Acceptance, Arrivals and Departures

JEL Classifications: C78, D47, D84

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[†]Department of Economics, Princeton University, Princeton, NJ 08544, (e-mail: muz@princeton.edu).

[‡]School of Economics and Management, Tsinghua University, Beijing, China, 100084 (e-mail: jie.academic@gmail.com).

1 Introduction

"I (__), take you, (__), to be my lawfully wedded (husband /wife), to have and to hold, from this day forward, for better, for worse, for richer, for poorer, in sickness and in health, until death do us part."

As suggested by the famous wedding vow, many multi-period matching markets in the real world require commitment to the current partner to some extent when the partnership is intended to last for the long term, which restricts agents' ability to unilaterally terminate a relationship. There are several typical examples as follows:

- **Kidney exchange – Matching with Full Commitment**: Kidney exchange has attracted much attention of researchers (e.g. Ünver, 2010[25]; Doval, 2015[7]) in terms of dynamics since naturally patients and kidneys arrive sequentially as time passes by. Another key feature of this economy is that the matching is irreversible in spite of the multi-period setting. That is, once a renal transplantation has been carried out, both the kidney and patient will permanently[1] leave the market. This once-and-for-all feature reveals that the paired agents will face full commitment and have to stick with the current matching forever.

- **Marriage market – Matching with Two-sided Commitment**: Due to the Hindu Marriage act, at least before 2013, a Hindu can obtain divorce only with mutual consent that requires one to complete at least the first anniversary of the marriage besides allowing a period of at least six months, which is intended to allow negative passions to cool down between the couple seeking divorce.[2] Similarly, effective October 1, 2015, mutual consent is a new (although not the unique) no-fault ground for absolute divorce in Maryland, US [3]. Actually, mutual consent reflects the idea of two-sided commitment in that paired agents can only terminate their relationship when both of them are weakly better under the separation.

- **School choice problem – Matching with One-sided Commitment**: Students may transfer or just drop out from the current school before graduation, and empirical evidence shows that they are actually excercising such a right. For example, Schwartz, Stiefel, and Chalico (2009)[21] shows that in the New York City primary schools, only 3.4% of 8th graders had attended the same school in the entire period from 1996-97 to 2000-01. Assume that students

[1]The word "permanently" may be a little misleading, since the patient may suffer from the disease again and need another kidney. The key (and reasonable) assumption here is that the patient typically does not take the possibility of exchanging a kidney again into account when deciding whether to accept the proposed match at present.

[2]Some are arguing for permission of unilateral divorce, see http://www.firstpost.com/investing/why-divorce-on-mutual-consent-doesnt-help-women-1124661.html. But at least between 1955 and 2013, only mutual consent divorce is allowed for Hindus.

[3]See https://www.peoples-law.org/no-fault-divorce.

and schools behave themselves,[4] then the school cannot require the present student to drop out in favour of a better candidate or to continue the enrollment. However, the student can freely choose to transfer or just drop out whenever she desires. A similar argument is applicable in situations where human beings are matched to services or positions such as insurances, as well as the labor market for tenured professors.

To acquire a quick impression of why commitment is essential to a dynamic matching market, one may consider the following example containing four agents.

Example 1.1 $M = \{m_1, m_2\}, W = \{w_1, w_2\}$ and the preferences (the preference symbol $\succ$ omitted):

$\mathbf{m_1}: w_1m_1 \quad w_1w_2 \quad w_1w_1 \quad m_1m_1;$	$\mathbf{m_2}: w_2w_1 \quad w_2w_2 \quad w_2m_2 \quad m_2m_2$
$\mathbf{w_1}: m_1m_2 \quad m_1m_1 \quad m_1w_1 \quad w_1w_1;$	$\mathbf{w_2}: m_2w_2 \quad m_2m_1 \quad m_2m_2 \quad w_2w_2$

With the concrete definitions in the paper, it is easy to show that the dynamically stable matchings with the four types of commitment – full commitment, two-sided commitment, one-sided commitment and no commitment – are all unique respectively and different from each other. For example, under full commitment, i.e, no breakups, the only dynamically stable matching is μ^{FC} where $\mu^{FC}(m_1) = w_1w_1$ and $\mu^{FC}(m_2) = w_2w_2$. However, μ^{FC} will be blocked by m_1 with $\mu'(m_1) = w_1m_1$ when m_1 is assigned the ability to remain single in period 2, that is, when there is one-sided commitment favoring M or no commitment. Likewise, μ^{FC} will not be stable under two-sided commitment since all agents are strictly better off if they exchange their previously assigned parnters in period 2. Thus, without specifying the commitment level, the mechanism adopted by the centralized clearing house may fail to be dynamically stable, even if it functions excellently in some other market.

Our work builds upon the seminal Deferred Acceptance (DA) algorithm by Gale and Shapley (1962), which subsequently inspired a broad and influential literature on matching and applications. Among the existing literature, dynamic matching is a relatively new area of research. In this paper, we examine the existence of dynamically stable matchings under different levels of commitment. In order to show the existence of such matchings we develop new algorithms which build upon Gale and Shapley's classic DA algorithm.

Despite its practical prevalence and importance, the effect of commitment has gained limited explicit attention in the literature about general dynamic matching theory. For instance, in a recent working paper by Kadam and Kotowski (2015a)[9], agents are entitled to dropping out from the current relationship freely without any cost since they can period 2 block the matching if

[4] For simplicity, we ignore the situation where some agent disobeys the contract terms and concentrate our attention on the breakup of matches purely resulting from preferences.

$(\mu_1(i), i) \succ_i \mu(i)$, which suggests the presence of no commitment. One possible explanation is that the relationship will naturally expire after one period, which is not realistic in many cases such as the marriage market and school choice problems, where the relationship will go on until death or graduation without any external interventions. By comparison, the prior works which exhibit the characteristics of some level of commitment typically originate from some real-world matching problems, like full commitment in Ünver (2010)[25] or one-sided commitment in Kurino (2014)[14], where acceptability requries that all existing tenants find their current houses at least as desirable as their previous ones. In this paper, we study the role of commitment explicitly in a two-period one-to-one matching market. Such a model can be easily extended to arbitrarily many periods, as is suggested in Section 6.

We begin with the baseline model with full commitment. Intuitively, the market resembles the static market to some extent since individuals only need to care about when and who they are originally matched with. Thus, many results in the classical static problem have valid counterparts here without any additional assumptions. For instance, the existence of dynamic stability can be guaranteed with a modified version of Gale and Shapley's famous DA algorithm, which is called PDA-FC algorithm. Also, the PDA-FC algorithm is strategy-proof for the proposing side and any dynamically stable matching with full commitment is Pareto efficient. Additional results concerning the structure of the set of dynamically stable matchings are also proven. However, there is no dynamically stable spot rule, which only depends on spot markets, unless strong assumptions are imposed on preferences, such as agents being extremely impatient. This highlights the difference between dynamic and static models, even if there are no breakups.

We then discuss the market with two-sided commitment, where mutually agreed upon termination of relationships is permitted. With the assumption of rankability as used in Kennes, Monte and Tumennasan (2014a)[11], which means that each individual has a ranking of possible partners even if his or her preference is defined over complete partnership plans, the existence of dynamic stability and some version of efficiency and strategy-proofness for the proposing side can be proven. Nevertheless, since rankability is incompatible with arrivals and departures of agents in period two, we put forth several different ways to deal with this issue, such as adding restrictions on the blocking power of agents appearing only in period 2, assuming that engagement is also binding, or narrowing the class of preferences. Correspondingly, a set of mechanisms called triggered DA algorithms are utilized to achieve stability.

The case with one-sided commitment is similar in terms of the methodology used. Extra conditions are also necessary to study the change in the underlying set of agents. However, since the one-sided commitment typically features the matching between human beings and institutions, where humans are usually favored, we focus on the arrivals and departures of the set of students in the college admission problem and some asymmetric assumptions between the two sides are applied.

One key contribution of our study is that we introduce a triggered DA algorithm used to carry out mutually beneficial breakups and rematches without harming anyone involved. Specifically, in period 2, agent i on the proposing side matched in period 1 can make proposals for rematches only when her current partner has received some proposal and thus 'triggers' i. Such an algorithm serves as a complement to the well-known TTC algorithm, where only one side is guaranteed to be better off and the welfare of the other side is not taken into account.

Another contribution concerns the treatment of arrivals and departures in a finite-horizon model. Typically, changes in the set of market participants over time are studied in an infinite-horizon model (such as overlapping generations) as in Doval (2015)[7], which may not be applicable in some real-world examples. [5]. In Kadam and Kotowski (2015a[9], b[10]), dynamic stability or other important results are hard to maintain when there exist arrivals (agents who only appear in the later periods) and departures (agents who only appear in the earlier periods). This paper explicitly studies the difficulty of incorporating those agents as well as some possible ways to involve the changes in the set of agents without sacrificing stability.

The remainder of the paper proceeds as follows. Section 2 summarizes the related literature. Sections 3, 4 and 5 examine the models with three kinds of commitment and find assumptions and algorithms to produce the corresponding stability notions. Section 6 provides discussions for several important issues, including the extension of the two-period model to multiple periods. Section 7 concludes. Appendix A discusses some notions of stability and assumptions on preferences in the literature of dynamic matching. Appendix B contains the proofs.

2 Literature Review

Our study owes its foundation to the original classic work by Gale and Shapley (1962). Gale and Shapley's algorithm was designed for static matching markets, in the sense that they consider a fixed pool of agents on each side of the market. The multi-period matching market differs from the static one mainly in the underlying sets of agents and dynamics of agents' preferences, which further raises the question of how the individuals should be committed to the current partner. Our

[5] See section 5.4 for a more detailed discussion

study addresses this question by considering four levels of commitment (full, one-sided, two-sided, no commitment), defines concepts of dynamic stability, and shows whether and under what additional conditions a dynamically stable matching exists. In the process, we develop several modifications of the DA algorithm which allow us to demonstrate the existence of stable matches in the dynamic setting.

The idea of dynamic arrivals and departures is internally important in many markets where there exist agents unmatched and waiting for newcomers. There typically exists a trade-off between waiting for a thicker market and being matched immediately to avoid discounting and waiting expenses (Akbarpour, Li and Oveis Gharan, 2014)[1]. For example, Ünver (2010)[25] studies the dynamic kidney exchange economy where waiting is costly and puts forward dynamically efficient centralized mechanisms that maximize total exchange surplus. Baccara, Lee and Yariv (2015)[2] compare the welfare implications between the optimal centralized algorithm and the decentralized matching process where the latter one will result in longer queues and thus inefficiency. Doval (2015)[7] further considers the existence of dynamical stability in different markets based on the incentives of waiting. Other works concentrate on concrete mechanism designs for specific markets. For instance, Dimakopoulos and Heller (2014)[6] study the German entry-level labour market for lawyers, where lawyers are allowed to wait, and Thakral (2015)[24] considers the allocation problem of public housing, which arrives stochastically over time, to agents in a queue. All above models focus on once-and-for-all matching in the multi-period setting, that is, agents only exhibit stable preferences over the individuals (or group of individuals in many-to-one or many-to-many settings) on the other side, and thus they leave the market permanently once they get matched, which coincides with the idea of full commitment. Our study includes a representation of such a problem with full commitment, yet in a finite-horizon framework.

If we allow for divorces and rematches in the dynamic market, the preferences should be defined on partnership plans, which specify one's partners in every single period. Then the most urgent task is to develop a reasonable conceptual system to extend the notions of static stability. Damiano and Lam (2005)[5] discuss the limitations of core and recursive core, and then define (strict) self-sustaining stability which imposes more restrictions on potential deviating coalitions. Kurino (2009)[13] utllizes a different definition of credible group stability. Typically time-separate and time-invariant preferences are assumed, and Bando (2012)[3] further shows that for pairwise stability, the assumption can be relaxed to substitutability or history dependence. Furthermore, Kadam and Kotowski (2015a)[9] highlight that the common elements such as status-quo, switching costs and desire for variability have been excluded by any of the previous assumptions and thus put forward a new notion of dynamic stability, whose existence can be guaranteed by weaker conditions. Since our study considers different levels of commitment, different notions of dynamic stability are correspondingly needed to analyze the problem.

Our study is strongly related to Kadam and Kotowski (2015a)[9] in terms of questions and concepts. The most important conceptual difference between Kadam and Kotowski (2015a) and our model is the strength of commitment studied. As is standard in the literature, they implicitly assume that the partnership is unbinding, that is, any agent can freely withdraw from the existing relationship in later periods with zero cost [6], which is not prevalent in all situations given the restrictive power of existing relationships because of costly liquidated damages, binding laws or social norms. Thus, their model is classfied as the case with no commitment, which will be concisely summarized in Section 6.

By comparison, some recent research concerning some specific markets do (implicitly) capture commitment to a certain extent. For example, in the Danish daycare assignment problem (Kennes, Monte and Tumennasan, 2014a[11], b[12]), the public school will give the top priority to those who have attended it the last year, which means that students will only transfer voluntarily. Similarly, in Pereyra (2013)[15], teachers are entitled with the right to continue in the school to which they were assigned in the previous period. Also, in the dynamic house allocation problem studied in Kurino (2014)[14], acceptability requires that all existing tenants weakly prefer their current houses to their previous ones. All those works reflects one-sided commitment on the side of schools or houses, which typically possess priorities instead of preferences.

3 Baseline Model with Full Commitment

We first consider the multi-period matching market with full commitment, or permanent matchings, where agents leave the market immediately after being matched. For example, agents may have consistent preferences across time so that preferences can be defined over individuals rather than complete partnership plans indexed by time (e.g. kidney exchange, see Ünver, 2010). Alternatively, the matching is reasonably once-for-long or even once-and-for-all, and thus we do not need to consider breakup of established relationships in the economic scope of interest, such as child-adoption (Baccara, Lee and Yariv, 2015) and housing markets. Besides these practical instances, the case with full commitment can serve as an appropriate baseline for other extensions with limited commitment. Accordingly, we will start with the case of full commitment.

[6]In fact, their assumptions of preferences do induce a bias toward consistent plans and that is how their mechanism typically produces a dynamically stable matching with few divorces.

3.1 Model Setup

In a two-period one-to-one matching market, M_t (W_t), $t = 1,2$ defines the set of men (women) available to be matched in period t. Denote $W_t^m \equiv W_t \cup \{m\}$ and $M_t^w \equiv M_t \cup \{w\}$ for any $t = 1,2$ and $m \in M_t, w \in W_t$. Any agent $m \in M_1 \cap M_2$ ($w \in W_1 \cap W_2$) holds a strict and rational preference $\succ_m^0$ ($\succ_w^0$) over partnership plans $(x_1,x_2) \in W_1^m \times W_2^m$ ($(x_1,x_2) \in M_1^w \times M_2^w$), or x_1x_2 for convenience when confusion is unlikely. Similarly, agents only appearing in the market for one period will hold strict and rational spot (current) preferences as those on a static market. Denote $R = \{\succ_i^0 : i \in M_1 \cup M_2 \cup W_1 \cup W_2\}$. Then $\mu = (\mu_1,\mu_2) : M_1 \cup M_2 \cup W_1 \cup W_2 \to (M_1 \cup W_1) \times (M_2 \cup W_2)$ is a **multi-period matching** if $\mu_t : M_t \cup W_t \to M_t \cup W_t$ is a spot matching on the static market for period t and $\mu(i) = i$ if $i \notin W_t \cup M_t$. Again, for convenience, we refer to μ as a matching. Then (M_1,M_2,W_1,W_2,R) is a **two-period marriage market.**

Extended Marriage Market

An approximately equivalent way to treat arrivals and departures is by adjusting preferences. Denote $M = M_1 \cup M_2$, $W = W_1 \cup W_2$, $\overline{R} = \{\succ_i : i \in M_1 \cup M_2 \cup W_1 \cup W_2\}$ which will be specified as follows, then $(M,W,\overline{R})$ is an **extended marriage market** of (M_1,M_2,W_1,W_2,R) if

- For $m \in M$, $\succ_m$ is defined over $(W^m)^2$;
- For $m \in M_1 \cap M_2$, $\succ_m = \succ_m^0$ over plans in $W_1^m \times W_2^m$;
- For $m \in M_1 \backslash M_2$, $mm \succ_m x_1x_2$ if $x_2 \neq m$ and $y_1m \succ_m y_2m \iff y_1 \succ_m^0 y_2$ for $y_1,y_2 \in W_1^m$;
- For $m \in M_2 \backslash M_1$, $mm \succ_m x_1x_2$ if $x_1 \neq m$ and $my_1 \succ_m my_2 \iff y_1 \succ_m^0 y_2$ for $y_1,y_2 \in W_2^m$;
- Symmetric for $w \in W$

We then define the multi-period matching μ. Intuitively, in the extended market, all agents "appear" on the market in all periods and the actual absence is reflected in agents' preferences. For instance, when m arrives in period 2, he will regard any plan where he is not single in period 1 as unacceptable.

The extension is called an approximate equivalence for two reasons. Firstly, although the set of matchings has been diluted and the preferences $\overline{R}$ are not uniquely determined by R, their sets of individually rational matchings, which we are interested in, coincide since those apparently unreasonable matchings will never happen given voluntary participation and thus only R will actually influence the final outcomes. Secondly, when defining stability, the adjustment may induce some practical questions like "how an agent appearing in the second period participates in period-1 blocking?". Typically, the extended market does not distinguish between agents arriving at different periods and thus can only define identical blocking rules for all players, which ignores the aforementioned question. Whether we can simplify our notations by using $(M,W,\overline{R})$ to replace (M_1,M_2,W_1,W_2,R) depends on the notion of dynamic stability we use.

Def 1. (Dynamic Stability with Full Commitment, DSFC) *A matching μ on the market (M_1,M_2,W_1,W_2,R) is dynamically stable with full commitment if*

1. *μ satisfies full commitment, that is, if $\mu_1(x) \neq x$, then $\mu_1(x) = \mu_2(x)$, $\forall x \in M_2 \cup W_2$;* [7]

2. *μ is not blocked by any individual, that is, $\nexists x \in M \cup W$ such that $xx \succ_x \mu(x)$ if x stays on the market for two periods or $x \succ_x \mu_t(x)$ if x stays on the market for only period $t = 1$ or 2;*

3. *μ is not blocked by any pair of agents, that is, $\nexists(m,w)$ such that (i). $ww \succ_m \mu(m)$ and $mm \succ_w \mu(w)$;*
   or (ii). If $m \in M_1 \cap M_2, w \in W_1 \cap W_2$, $mw \succ_m \mu(m)$ and $wm \succ_w \mu(w)$; If $m \in M_2 - M_1, w \in W_1 \cap W_2$, $w \succ_m \mu_2(m)$ and $wm \succ_w \mu(w)$; If $m \in M_1 \cap M_2, w \in W_2 - W_1$, $mw \succ_m \mu(m)$ and $m \succ_w \mu_2(w)$; If $m \in M_2 - M_1, w \in W_2 - W_1$, $w \succ_m \mu_2(m)$ and $m \succ_w \mu_2(w)$;
   or (iii). $m \in M_1 - M_2, w \in W_1 - W_2$, $w \succ_m \mu_1(m)$ and $m \succ_w \mu_1(w)$.

Def 2. (Weak Dynamic Stability with Full Commitment, WDSFC) *A matching μ on the market (M_1,M_2,W_1,W_2,R) is weakly dynamically stable with full commitment if*

1. *μ satisfies full commitment, that is, if $\mu_1(x) \neq x$, then $\mu_1(x) = \mu_2(x)$, $\forall x \in M_2 \cup W_2$;*

2. *μ is not blocked by any individual, that is, $\nexists x \in M \cup W$ such that $xx \succ_x \mu(x)$ if x stays on the market for two periods or $x \succ_x \mu_t(x)$ if x stays on the market for only period $t = 1$ or 2;*

3. *μ is not blocked by any simultaneous pair of agents, that is, $\nexists(m,w)$ such that (i). $m \in M_1 \cap M_2, w \in W_1 \cap W_2$, $ww \succ_m \mu(m)$ and $mm \succ_w \mu(w)$ or $mw \succ_m \mu(m)$ and $wm \succ_w \mu(w)$;*
   or (ii). $m \in M_2 - M_1, w \in W_2 - W_1$, $w \succ_m \mu_2(m)$ and $m \succ_w \mu_2(w)$;
   or (iii). $m \in M_1 - M_2, w \in W_1 - W_2$, $w \succ_m \mu_1(m)$ and $m \succ_w \mu_1(w)$.

We can also define fairness or justified envy-freeness in a similar way for markets where dynamic stability is not the main concern.

The notion of dynamic stability with full commitment is a natural extension of stability in the static market to a multi-period one, under the restriction of once-and-for-all matching. Here everyone

[7] Actually, full commitment only prevents agents staying on the market for two periods from breaking up if they are matched in period 1. However, agents can terminate the partnership plan if they both leave the market in period 2.

possesses the same blocking power and thus the extended marriage market is justified. By comparison, the weak stability corresponds to the practical question concerning the feasibility of future agents being involved in the blocking of current matching. Note that players in $M_2 \backslash M_1$ can only form a blocking pair with someone unmatched in the first period, because by the time he appears those who are matched previously have left the market. In this way, agents arriving at different periods have asymmetric blocking powers. Although the former definition of stability is stronger, it is unclear whether it suits better for all markets. In the main part of the paper, we will focus on dynamic stability with full commitment(DSFC) and results on weak dynamic stability (WDSFC) are discussed in Section 6. For simplicity, we use the extended marriage market for illustration in this section.

Moreover, in any stable matching with no divorce allowed, those who leave the market in the second period will either remain single for two periods or be matched with someone who also stays in the market only in the first period, since our definition of full commitment requires an agent being matched to the same partner for both periods (note that one's partner exiting the market is treated in our setup as a type of divorce). Actually, in the general once-for-all matching market (Akbarpour, Li and Oveis Gharan, 2014; Baccara, Lee and Yariv, 2015; Thakral, 2015; Doval, 2015), agents will typically remain in the market until they get matched or their patience runs out. Thus, without loss of generality, we can implicitly assume that $M_1 \subset M_2, W_1 \subset W_2$ as those in $(M_1 \backslash M_2) \cup (W_1 \backslash W_2)$ will are considered irrelevant agents and can be excluded from the market of our interest.

3.2 Existence of Stability

Theorem 1 The set of dynamically stable matchings with full commitment is nonempty for any extended marriage market $(M, W, \overline{R})$ (and thus in the original market $(M_1, M_2, W_1 \cup W_2, R)$).[8]

We will prove the existence by providing an algorithm called plan deferred acceptable algorithm with full commitment (P-DAFC), a modification from the famous Gale-Shapley defered acceptance algorithm. Under the restriction of full commitment, men will only propose to a woman with a consistent plan (*ww*) or engagement (*mw*) and the remaining algorithm proceeds as in the static counterpart.

Algorithm 1 (PDA-FC) The man-proposing plan deferred acceptable algorithm identifies a matching μ^* as follows:

[8] All the proofs from now on are deferred to the appendix.

1). For each $m \in M$, let $X_m^0 \equiv \{ww, mw, mm : w \in W\}$. Initially, no plans in X_m^0 have been rejected;
2). In round $\tau \geqslant 1$,
(a). Let $X_m^\tau \subset X_m^{\tau-1}$ be the set of plans that have not been rejected in the previous rounds and m propose the most preferred plan in X_m^τ. Proposing mm implies that agent m has been rejected by any acceptable plans involving a woman.
(b). Let X_w^τ denote the set of plans proposed to w in round τ. If $ww \succ_w x_1x_2$ for all $x_1x_2 \in X_w^\tau$, then w rejects all the proposals. Otherwise, w keeps her most preferred plan in X_w^τ tentatively and rejects all others.
3). Repeat procedure 2) until no further rejections occur. If w ends up keeping m's proposal in the final round, define $\mu^*(m)$ and $\mu^*(w)$ accordingly. If i does not make or keep any proposal in the final round, let $\mu^*(i) = ii$.

3.3 Properties of Dynamic Stability with Full Commitment

Based on the similarity of P-DAFC and static DA algorithms, we now want to consider whether the nice properties about stability in classical one-period markets have similar versions in the multi-period setting. In contrast to dynamic stability with no commitment (KK, 2015a) where such an analogy fails without restrictive assumptions on preferences, full commitment maintains the intuition of static markets as much as possible and thus the validity of many nice results is preserved.

Theorem 2. (Mutual Interests on the Same Side) Let $(M, W, \overline{R})$ be a two-period marriage market in which all preferences are strict, then the M-optimal (W-optimal) dynamically stable matching with full commitment exists, and coincides with the outcome of corresponding P-DAFC μ^M (μ^W).

Denote the set of dynamically stable matchings with full commitment as S. Define a plan jk as **achievable** for agent i if $\exists \mu \in S$ such that $\mu(i) = jk$. Then the intuition just follows the theorem in the static market such that in the P-DAFC, no agents on the proposing side have any achievable plan rejected by someone on the receiving side and thus by simply assigning each man to his most preferred achievable plan we can produce a stable matching μ^* as suggested by the P-DAFC.

Theorem 3. (Conflicting Interests on Opposite Sides) Let $(M, W, \overline{R})$ be a two-period marriage market in which all preferences are strict, and $\mu^1, \mu^2 \in S$ as two stable matchings with full commitment. Then $\mu^1 \succ_M \mu^2 \Longleftrightarrow \mu^2 \succ_W \mu^1$.

Intuitively, the above algorithm implies that under the condition of stability, a change that makes the set of men better off will inevitably hamper the benefits of the set of women. There are two natural corollaries concerning the worst stable matching and the equivalent condition for uniqueness.

Corollary 1. Let $(M, W, \overline{R})$ be a two-period marriage market in which all preferences are strict, and the M-optimal (W-optimal) dynamically stable matching with full commitment is the W-pessimal (M-pessimal) dynamically stable matching with full commitment.

Corollary 2. (Uniqueness) Let $(M, W, \overline{R})$ be a two-period marriage market in which all preferences are strict, then there is a unique dynamically stable matching with full commitment, that is, $|S| = 1$ if and only if $\mu^M = \mu^W$.

Strategy-proofness has been regarded as a key issue in the literature of mechanism design not only because it is hard to model agents' strategic behaviour completely due to the complexity of higher-order beliefs but also because of the potential costs induced by playing games with others, especially in the case of school choice and public housing allocation where fairness is typically a major concern. The argument of Roth (1982) shows the nonexistence of a strategy-proof stable matching mechanism on both sides of the market, and thus the attention is drawn to concentrate on the strategic behaviour on one side (See Chapter 4, Roth and Sotomayor, 1990). In fact, in many practical markets like allocation problems, the preference or priority ranking on one side of the market is common knowledge or determined by the social planner who only cares about fairness and strategy-proofness, leaving no room for manipulations on this side. Examples include organs in the kidney exchange, rooms in the housing allocation problem and schooling positions in the school choice problem. Thus one-sided strategic proofness is also practically relevant.

Theorem 4. (Strategic Issues)

(1). No stable matching mechanism with full commitment exists for which stating the true preferences is a (weakly) dominant strategy for every agent;

(2). Let $(M, W, \overline{R})$ be a two-period marriage market in which all preferences are strict, then the M-proposing P-DAFC makes it a (weakly) dominant strategy for each man to report his true preference. Moreover, no coalition of men can improve the outcome for all members in the coalition by misreporting under M-proposing P-DAFC. Symmetric for the W-proposing P-DAFC.

Efficiency is another important criterion for matching markets of our major concern, especially in terms of social welfare and centralized market design. Fortunately, when preferences are strict, every dynamically stable matching with full commitment is Pareto optimal. Intuitively, if there is a feasible Pareto improvement, the agent who becomes better off can find an agent on the other side to form a blocking coalition of the original matching, based on strictness of preferences. Moreover, the matching produced by the deferred acceptance algorithm is weakly Pareto efficient for the proposing side, which partially contributes to the result of one-sided strategyproofness. The intuition lies in the fact that the DA algorithm will terminate as soon as everyone on the receiving side is holding an acceptable proposal.

Theorem 5. (Efficiency)
(1). Let $(M,W,\overline{R})$ be a two-period marriage market in which all preferences are strict, then every dynamically stable matching with full commitment is Pareto efficient within the set of matchings with full commitment.
(2). Furthermore, there does not exist any individually rational matching with full commitment μ s.t. $\mu \succ_m \mu^M$, $\forall m \in M$. Similarly, there does not exist any individually rational matching with full commitment μ s.t. $\mu \succ_w \mu^W$, $\forall w \in W$.

Lastly, we want to look into the structure of the set of dynamically stable matchings with full commitment. For instance, can we derive a third DSFC matching from two existing ones?

Def 3. (Meet and Join of Two Matchings) *Let $(M,W,\overline{R})$ be a two-period marriage market in which all preferences are strict and μ, μ' be two matchings. Define $\lambda \equiv \mu \vee_M \mu'$ as the join of μ and μ' for M if λ is a function $M \cup W \to (M \cup W)^2$ such that:*
(1). $\forall m \in M$, $\lambda(m) = \mu(m)$ if $\mu(m) \succ_m \mu'(m)$ and $\lambda(m) = \mu'(m)$ if $\mu'(m) \succeq_m \mu(m)$;
(2). $\forall w \in W$, $\lambda(w) = \mu'(w)$ if $\mu(w) \succ_w \mu'(w)$ and $\lambda(w) = \mu(w)$ if $\mu'(w) \succeq_w \mu(w)$.
We can also define $\gamma \equiv \mu \wedge_M \mu'$ as the meet of μ and μ' for M in the opposite way

Intuitively, the join of two matchings for M assigns each man his better plan among the two possible choices and assigns each woman her worse plan, while the meet for M assigns each man his worse plan and each woman her better plan. By definition, we have $\vee_M = \wedge_W$ and $\vee_W = \wedge_M$. For general matchings, the join or meet is not necessarily a matching, let alone stable; but if μ and $\mu' \in S$, the transformation will also produce a stable matching with full commitment. This further specifies the extent to which the common interests on one side coincide with the conflicting interests on opposite sides.

Theorem 6. (Lattice structure) Let $(M,W,\overline{R})$ be a two-period marriage market in which all preferences are strict, then the set of dynamically stable matchings with full commitment forms a lattice under $\succ_M$ or $\succ_W$.

4 Matching with Two-sided Commitment

We now consider one extension of the baseline model with full commitment to allow for agreed divorce in our two-period matching market, that is, the couple can break up peacefully when both of them are willing to do so. However, if divorce can only engender benefits to one of the couple, then it is still assumed impossible. This seems like the opposite extreme situation of the no-commitment case in KK (2015a, b) as the cost of divorce without mutual agreement is changed from 0 to infinity. Meanwhile, this modification in the set up also calls for adjustment in the notion of stability. For example, now that the marriage is intended to last for long, both partners in the couple can force the current relationship to persist, which contradicts the KK (2015a) notion that agents can freely break the promise and remain unmatched in period 2.

Stability with two-sided commitment corresponds to the idea of mutual-consent (no-fault) divorce in legal practice. As mentioned in the introduction, we observe several examples of mutual-consent uncoupling in reality, such as the Hindu Marriage Act, and mutual consent divorce in Maryland.

As for the literature, Sun and Yang (2016)[23] studies mutual consent divorce in a marriage market with some fixed ex-ante matching. Married couples can divorce and thus remarry as long as both parties will not be made worse off than if they maintain the status quo. Actaully, this is still a static model since the first-period matching can be regarded as exogenously given. By comparison, we endogenize the formation of matchings in both periods. In other words, if the matching in period one is determined, the two models coincide; however, under complete information and backward induction, period-one relationships will depend on the matching prospects in the future. This interaction between spot matchings at different times distinguishes our work from Sun and Yang (2016).

The basic setup borrows from markets with full commitment. To begin with, we define a new behavioral assumption on preferences used by Kennes, Monte and Tumennasan (2014a, b):

Def 4. (Rankability) *For $m \in M$, $\succ_m$ satisfies rankability if $\forall x_1, x_2 \in W \cup \{m\}, x_1x_1 \succ_m x_2x_2 \Longleftrightarrow yx_1 \succ_m yx_2$ and $x_1y \succ_m x_2y$, $\forall y \in W \cup \{m\}$ and $y \neq x_2$. Similarly we can define $\succ_w$ for $w \in W$.*

Intuitively, rankability implies that the preference over complete plans is closely related to some spot preferences over individuals. Special value on variation in partners is ruled out but agents may be asymmetrically fond of consistency as it is required that $y \neq x_2$ in the definition. Thus, it is possible that $w_1w_1 \succ_m w_2w_2 \succ_m w_1w_2$, which means a long-run relationship with a mediocre partner may be better than an inconsistent plan involving both medicore and great partners. Such an assumption is commonly used in the literature on dynamic matching. For example, another version of the assumption called 'inertia' is proposed by KK(2015a, b), which can be shown to be equivalent to rankability (See Appendix A). Similarly, in Pereyra (2013), each teacher has preferences defined over a single period, which are time-separable and time-invariant, and thus satisfy rankability.

However, rankability is not compatible with essential arrivals and departures if all agents' preferences are defined over the same time horizon. Specifically, consider the case where everyone cares about matching in period one and two. m joins the market in period 2, and $mm \succ_m ww$ for any $w \in W$. By rankability, we have $mm \succ_m mw$ for any $w \in W$. Thus, m will regard all plans except for mm as unacceptable and will definitely stay single all the time. Likewise, agents planning to leave in period 2 will also be unmatched for two periods. Accordingly, such arrivals and departures are inessential in that they can be ex-ante regarded to be absent from the market without changing the final outcome of any mechanism. In other words, the market proceeds as if there are no departures or arrivals. Such a phenomenon is implicitly reflected in the literature. KK(2015a, b) studies the repeated marriage market with a fixed set of participants. By comparison, Kennes, Monte and Tumennasan (2014a, b) considers a multi-period daycare problem over overlapping generations, where agents arriving in different periods are faced with different relevant time horizon. For instance, agents arriving in period one have preferences over partnerships in period one and two, while those arriving in period two only care about partners in period two and three. Moreover, schools only have priorities over school-age children that are determined by local municipality, instead of preferences. In this way, the conflicts between changes in the set of agents and rankable preferences are resolved.

In this paper, we allow for arrivals and departures by dividing the set of agents into three types:

- **Type 1**: Type-1 agents stay in the market for two periods and have rankable preferences. The set is denoted as $T_1 \subset M \cup W$;
- **Type 2**: Type-2 agents are those who (practically) enter the market in period 2 and thus regard all plans where he or she is matched in period 1 as unacceptable. The set is denoted as $T_2 \subset M \cup W$;
- **Type 3**: Type-3 agents are those who (practically) leave the market in period 2 and thus regard all plans where he or she is matched in period 2 as unacceptable. The set is denoted as $T_3 \subset M \cup W$; [9]

[9]Typically, $T_1 = (M_1 \cap M_2) \cup (W_1 \cap W_2)$, $T_2 = (M_2 - M_1) \cup (W_2 - W_1)$ and $T_3 = (M_1 - M_2) \cup (W_1 - W_2)$. However,

That is, only agents active on the market for two periods satisfy rankability. Relabel the market as $(M,W,(T_1,T_2,T_3),\overline{R})$.

For stability, a matching μ is **blocked by an individual i** if either (i) $ii \succ_i \mu(i)$ (first period blocking), or (ii) $(\mu_1(i),\mu_1(i)) \succ_i \mu(i)$ (second period blocking). μ is **individually rational (IR)** if it is not blocked by any individual. A matching μ is **period-1 blocked by a pair of agents** (m,w) if either (i) $ww \succ_m \mu(m)$ and $mm \succ_w \mu(w)$ or (ii). $mw \succ_m \mu(m)$ and $wm \succ_w \mu(w)$ or (iii). $wm \succ_m ww$, $wm \succ_m \mu(m)$, $mw \succ_w mm$ and $mw \succ_w \mu(w)$. Condition (iii) may seem strange as we impose the restriction $mw \succ_m ww$ and $wm \succ_w mm$ to the ordinary definition. As has been discussed above, agents know and can rationally expect that if they are matched in period 1, then their partner can prevent divorce unilaterally.

- If $ww \succ_m \mu(m)$ and $mm \succ_w \mu(w)$, the situation falls into category (i);
- If $ww \succ_m wm \succ_m \mu(m)$, $mw \succ_w \mu(w) \succ_w mm$, then agent w rationally expects that if she chooses to form a blocking pair with m, then the blocking matching should be $\mu'(w) = mm$ instead of $\mu'(w) = mw$ since agent m will always force the marriage to persist in period 2, which is worse than w's present partnership plan. In this way, w is unwilling to carry out the blocking pair with m. Thus we exclude such a kind of blocking.
- If $wm \succ_m \mu(m) \succ_m ww$, $mm \succ_w mw \succ_w \mu(w)$, the argument is the same as the above case;
- If $wm \succ_m ww$, $wm \succ_m \mu(m)$, $mw \succ_w mm$ and $mw \succ_w \mu(w)$, we permit the blocking to happen.[10]

Now it is time to define period-2 blocking. Here we use the language of core as in Sun and Yang (2016)[23] since the pairwise blocking may seem a little strange when incorporating the idea of peaceful divorce and this modification will only makes it easier for a blocking to happen, which will strengthen the validity of our results.

Def 5. (Period-2 Blocking by a Coalition) *$S \in M \cup W$ can period-2 block a matching μ with two-sided commitment if*

there may exist someone who stays in the market for two periods but acts as if she only enters in period 2 for some reason (such as an age restriction). Such agents are classified into Type 2.

[10] This is similar to the notion of self-sustaining recursive core in Damiano and Lam (2005). Or put in another way, if we do not make such an adjustment for condition (iii), then the existence of stability may fail to hold for markets with departures. For instance, consider $W_1 = \{w\}$, $W_2 = \emptyset$, $M_1 = M_2 = \{m\}$ and $ww \succ_m \succ_w m \succ_m mm$, $mw \succ_w ww$. For a matching μ to be individually rational, $\mu(w) = ww$ or mw. If $\mu(w) = ww$, then (m,w) period-one blocks μ via $\mu'(m) = wm$. If $\mu(w) = mw$, then m will period-2 blocks μ by enforcing the period-1 matching to persist. Thus, no dynamically stable matching with two-sided commitment exists in such a simple market.

i *Mutual involvement:* $\forall x \in S$, $\mu_1(x) \in S$;

ii *Implementation and mutual benefits:* $\exists\, \overline{\mu}_2 : S \to S$, *s,t* $\forall x \in S$, $\overline{\mu}_2(x) \in S$ *and* $(\mu_1, \overline{\mu}_2) \succ_S \mu$.

That is, a coalition can period-2 block a given matching if and only if the condition of peaceful divorce is satisfied and everyone can be weakly better off (with at least one agent strictly better off) by implementing a period-2 spot matching among S. We do not require that all agents strictly benefit from the blocking and thus some agents in the blocking set may be indifferent between the two matchings. However, their presence is nontrivial since it reflects the requirement of mutual involvement and two-sided commitment. Now we can define the corresponding notion of stability.

Def 6. (Stability with Two-sided Commitment, DSTC) *In a two-period marriage market* $(M, W, \overline{R})$, *a matching* μ *is* ***dynamically stable with two-sided commitment*** *if it is not blocked by any individual, not period-1 blocked by any pair of agents and not period-2 blocked by any coalition.*

4.1 Existence of Stability

Now it is time to examine the existence of DSTC matchings. Unfortunately, unlike the case with full commitment, there may be no DSTC matching under the general class of preferences.

Theorem 9. The set of dynamically stable matchings with two-sided commitment may be empty for some marriage market.

A counterexample is shown as follows:
Example 4.1 $M = \{m_0, m_1, m_3\}, W = \{w_0, w_1, w_2, w_3\}$ with $T_1 = \{m_1, w_1, m_3, w_3\}$, $T_2 = \{m_0, w_0, w_2\}$ and the preferences (the preference symbol $\succ$ omitted):

$\mathbf{m_0}: m_0w_1 \quad m_0w_2 \quad m_0m_0$; $\qquad$ $\mathbf{m_3}: w_2w_2 \quad w_3w_2 \quad m_3w_2 \quad w_3w_3 \quad m_3m_3$

$\mathbf{m_1}: w_3w_3 \quad w_3w_0 \quad w_3w_1 \quad w_1w_3 \quad w_0w_0 \quad w_1w_0 \quad w_1w_1 \quad m_1m_1$

$\mathbf{w_0}: w_0m_1 \quad w_0w_0$; $\qquad$ $\mathbf{w_2}: w_2m_0 \quad w_2m_3 \quad w_2w_2$

$\mathbf{w_1}: m_0m_0 \quad m_1m_0 \quad m_1m_1 \quad w_1w_1$; $\qquad$ $\mathbf{w_3}: m_3m_3 \quad m_1m_1 \quad w_3w_3$

Suppose that μ is DSTC, then there are three possible partnership plans for the individual rationality of m_0.

(1) If $\mu(m_0) = m_0m_0$, then $w_2m_0 \succ_{w_2} \mu(w_2)$ and $m_0w_2 \succ_{m_0} \mu(m_0) = m_0m_0$, and thus (m_0, w_2) period-1 blocks μ via $\mu'(m_0) = m_0w_2$. No DSTC matching exists in this case.

(2) If $\mu(m_0) = m_0w_2$, then $\mu(w_2) = w_2m_0$ by IR constraint of w_2. Then $\mu_2(m_3) \neq w_2$. To avoid

that (m_3, w_3) period-1 blocks μ, $\mu(w_3) = m_3m_3$. This implies $\mu_1(m_1) \neq w_3, \mu_2(m_1) \neq w_3$. Also, by IR of w_0, $\mu(m_1) \neq w_0w_0$. Then $\mu(m_1) = w_1w_1$ or m_1m_1 or w_1w_0.

(I) If $\mu(m_1) = w_1w_0$, then $\mu_1(w_1) = m_1$ but $\mu(w_1) \neq m_1m_1$. By IR of w_1, $\mu(w_1) = m_1m_0$ which is a contradiction as $\mu_2(m_0) = w_2 \neq w_1$.

(II) If $\mu(m_1) = m_1m_1$, then $\mu(w_1) = w_1w_1$ and (m_1, w_1) period-1 blocks μ.

(III) If $\mu(m_1) = w_1w_1$, then $\mu(w_1) = m_1m_1$, $\mu(w_0) = w_0w_0$ and $\mu(m_0) \neq m_0w_1$. Then $\{m_0, w_0, m_1, w_1\}$ period-2 blocks μ via $\bar{\mu}_2(m_0) = w_1$, $\bar{\mu}_2(w_0) = m_1$. Thus, μ is not DSTC.

(3) If $\mu(m_0) = m_0w_1$, then $\mu_2(w_1) = m_0$ and $\mu_1(w_1) \neq m_0$. By IR of w_1, $\mu(w_1) = m_1m_0$. For m_1, $\mu_1(m_1) = w_1 \neq \mu_2(m_1)$, by IR of m_1, $\mu(m_1) = w_1w_0$ or w_1w_3. If $\mu(m_1) = w_1w_3$, then $\mu_2(w_3) = m_1 \neq \mu_1(w_3)$, which implies that $\mu(w_3)$ is unacceptable for w_3, a contradition! If instead $\mu(m_1) = m_1w_0$, then for w_2, either $\mu(w_2) = w_2w_2$ or w_2m_3.

(I) If $\mu(w_2) = w_2w_2$, then $\mu_2(m_3) \neq w_2$, which means that $m_3w_2 \succ_{m_3} \mu(m_3)$. Also $w_2m_3 \succ_{w_2} w_2w_2$, thus μ is period-1 blocked by (m_3, w_2).

(II) If $\mu(w_2) = w_2m_3$, then $\mu_2(m_3) = w_2 \neq \mu_1(m_3)$. By IR of m_3, $\mu(m_3) = w_3w_2$ or m_3w_2. If $\mu(m_3) = w_3w_2$, then $\mu_1(w_3) = m_3 \neq \mu_2(w_3)$, which contradicts with IR of w_2. If $\mu(m_3) = m_3w_2$, then $\mu(w_3) \neq m_3m_3 or m_1m_1$ and thus $\mu(w_3) = w_3w_3$. So (m_1, w_3) will period-1 block μ via $\mu'(m_1) = w_3w_3$ and thus μ is not DSTC.

The above argument shows that all matchings in Example 4.1 can be somehow blocked. The key reason for this nonexistence result is that $m_3w_2 \succ_{m_3} w_3w_3$. Intuitively, when matched with w_3w_3, m_3 makes sure that he would not be worse off if any period-2 blocking occurs because of two-sided commitment. Meanwhile, he does prefer to form an engagement with the type-2 agent w_2. However, when $\mu(m_3) = m_3w_2$, w_2 can freely join a period-2 blocking set, leaving m_3 alone since engagement is assumed to be not restrictive in the above definition of DSTC. Accordingly, to guarantee the existence of stability, more restrictions should be imposed on the market studied. Practically speaking, three methods are possible.

4.1.1 Restriction on Period-1 Blocking: Limited Blocking Power of Entries

Recall that in the case with full commitment, we use the extended marriage market as a substitute for the original market since the blocking rule does not distinguish agents arriving (leaving) at the market in different times in the definition of dynamic stability with full commitment. Nevertheless, as we have argued, there may emerge a question such as ”how can an agent appearing in the second period participate in period-1 blocking?"

For justification, consider a decentralized framework. On the one hand, agents 'born' to the market in period 2 know nothing about the previous market until the end of the first period, when the period-1 matching μ_1 has already been determined. Thus, they cannot form a period-1 blocking pair with those who have been involved in a relationship with two-sided commitment. On the other hand, type-1 agents' preferences about type-2 agents may have not been well defined until their appearance. For instance, according to Roth (1984)[17], the labor market for medical interns between hospitals and graduates from medical schools in the US exhibited serious advancements in the determination of appointments and agreements as compared to the actual starting date of the internship. To deal with the unravelling problem, the Association of American Medical Colleges (AAMC) adopted a proposal that neither scholastic transcipts nor letters of reference would be released prior to the end of the junior year for students seeking internships commencing in 1946. Hospitals can only know their specific preferences over students when they arrive in the labor market, ready to take a position. Thus, the hospital will not period-1 block the matching with an agent whose information is revealed in period 2.

Similarly, in a centralized framework, the social planner cannot require type-2 agents, who have not appeared on the market in period one, to report their preferences ex ante. Thus, she has to determine μ_1 without any information about type-2 agents. In this way, the planner may desire a matching without any justified envy (any blocking pair), but can only manage to enforce a sub-optimal outcome where period-1 blocking between type-2 agents and matched type-1 agents still exists.

Moreover, another possible reason for limited blocking power of arrivals is that agents may not possess a commitment device to enforce engagements or an unravelling contract. For example, agents may be skeptical that the lack of binding engagements makes it possible for their partners to betray the pre-determined matching in the following periods and thus refuse to take the risk. Likely, another way to deal with the above unravelling problem in the labor market for medical interns is to prevent individuals from forming binding agreements too early by official regulations.

By 'limited blocking power of entries', we mean that type-2 agents cannot form a period-1 blocking pair with type-1 agents who are matched in the first period. Formally, we have the following definition of weak dynamic stability with two-sided commitment.

Def 7. (Weak Dynamic Stability with Two-sided Commitment, WDSTC) *In a two-period mar-*

riage market $(M,W,(T_1,T_2,T_3),\overline{R})$, a matching μ is ***weakly dynamically stable with two-sided commitment*** *if it is not blocked by any individual, not period-1 blocked by any pair of agents among $T_1 \cup T_3$ or among $T_2 \cup T_3$ or among $T_2 \cup (T_1 \cap \{i : \mu_1(i) = i\})$, and not period-2 blocked by any coalition.*

Fortunately, the existence of WDSTC matching is guaranteed.

Theorem 7. The set of weakly dynamically stable matchings with two-sided commitment is nonempty for any marriage market $(M,W,(T_1,T_2,T_3),\overline{R})$.

To prove the above existence theorem, we propose an algorithm that can produce a WDSTC matching for any market. Before going into the mechanism, we firstly define the spot preferences.

Def 8.(Spot Preference)

1. For $m \in M$, $\succ_m^1$ defined over W^m is the spot preference in period 1 for agent m if $x \succ_m^1 y \Longleftrightarrow xx \succ_m yy$, $\forall x,y \in W^m$.

2. For given μ_1, $m \in M$, $\succ_m^{2(\mu_1)}$ defined over W^m is the spot preference in period 2 for agent m if $x \succ_m^{2(\mu_1)} y \Longleftrightarrow (\mu_1(m),x) \succ_m (\mu_1(m),y)$, $\forall x,y \in W^m$.

3. w's spot preferences is defined correspondingly.

Algorithm 2 (Two-Stage T-DA) The man-proposing **two-stage triggered deferred acceptance algorithm** identifies a matching μ as follows:

- *Stage 1. Restricted P-DA:*
  Consider a two-period marriage market $(M,W,(T_1,T_2,T_3),\overline{R})$. Define $\overline{R}_1$ as the preferences restricted to certain plans, that is, for agent $m \in T_1$, $\overline{R}_1(m)$ agrees with $\overline{R}(m)$ on $X_m^1 \equiv \{ww,mm : w \in W\} \cup Y_m^1 \equiv \{wm : w \in W \text{ and } mw \succ_w mm\} \cup Z_m^1 \equiv \{mw : w \in (W - T_2)\}$ and all other plans are regarded as unacceptable.
  For agent $m \in T_2$, $\overline{R}_1(m)$ agrees with $\overline{R}(m)$ on $X_m^1 \equiv \{ww,mm : w \in W\} \cup Y_m^1 \equiv \{wm : w \in W$ and $mw \succ_w mm\} \cup Z_m^1 \equiv \{mw : w \in (W - T_1)\}$ and all other plans are regarded as unacceptable.
  For agent $m \in T_3$, $\overline{R}_1(m)$ agrees with $\overline{R}(m)$ on $X_m^1 \equiv \{ww,mm : w \in W\} \cup Y_m^1 \equiv \{wm : w \in W$ and $mw \succ_w mm\} \cup Z_m^1 \equiv \{mw : w \in W\}$ and all other plans are regarded as unacceptable.
  Similarly we can define $\overline{R}_1(w)$. Then run the P-DA, which is modified from P-DAFC with $X_m^1 \cup Y_m^1 \cup Z_m^1$ replacing X_m^0 as potential proposed plans for the market $(M,W,\overline{R}_1)$ and get the interim matching μ^I.[11]

[11] By comparison, in P-DAFC, the proposing side can only propose plans that satisfy full commitment (consistency); here, *m* can also choose plans in the form of *wm*, but type-1 agents and type-2 agents will not propose to each other

- *Stage 2. M-proposing Triggered DA for Period 2:*
  For notation, denote $D^M = \{m : \mu_1^I(m) = m\} - T_3$, $D^W = \{w : \mu_1^I(w) = w\} - T_3$, $D^P = \{i : \mu_1^I(i) \neq i\} - T_3$. Here we exclude type-3 agents since they will definitely be unmatched in period 2 and their partnerships have been fully determined by μ^I. Denote $E^m = \{w \in D^P \cup D^W : (\mu_1^I(m), w) \succsim_m (\mu_1^I(m), \mu_1^I(m))\ \textit{and}\ (\mu_1^I(w), m) \succsim_w (\mu_1^I(w), \mu_1^I(w))\}$ as the set of women who can mutually benefit from being matched with m in period 2 with respect to μ^I. Moreover, given the period-2 spot preferences $\succ_w^{2(\mu_1^I)}$, agent w's priority is determined as follows: for $w \in D^W$, the priority is the same as $\succ_w^{2(\mu_1^I)}$; for $w \in D^P$, $\mu_1^I(w)$ has the highest priority and the priority for others coincides with $\succ_w^{2(\mu_1^I)}$.

  - *Round 1*: Each man $m_i^2 \in D^M$ proposes to his most preferred agent in $E^{m_i^2}$ (if any);
    If some $w_i^1 \in D^P$ receives a proposal, then her period-1 partner $\mu_1^I(w_i^1) = m_i^1$ is 'triggered' and will propose to his most preferred agent in $E^{m_i^1}$ (if any);
    Each $w \in D^W \cup D^P$ holds the acceptable proposal with the highest priority (if any) and rejects all others.
    … …
  - *Round k ($\geqslant 2$)*: Each man m rejected in the $(k-1)th$ round proposes to his most preferred agent in $E^m \cap \{w : w \text{ has not rejected } m\}$ (if any);
    If some $w_i^1 \in D^P$ who has not been proposed to in the previous rounds receives a proposal, then her period-1 partner $\mu_1^I(w_i^1) = m_i^1$ is 'triggered' and will propose to his most preferred agent in $E^{m_i^1}$ (if any);
    Each $w \in D^W \cup D^P$ holds the acceptable proposal with the highest priority among the new offers and the one she held in the last round(if any), and rejects all others.

    … …
  - *Outcome*: The algorithm terminates when no more rejections happen. μ_2^{II} matches each m to w who is holding his offer (if any) and for all others, $\mu_2^{II}(i) = \mu_2^I(i)$.

  $\mu = (\mu_1^I, \mu_2^{II})$ is the ultimate outcome of the man-proposing **two-stage triggered deferred acceptance algorithm**.

There are several observations worth mentioning about the above algorithm. Firstly, if some man m is matched in period 1, then he can only have the opportunity to propose for a profitable rematch in period 2 after his period-1 partner $w = \mu_1^I(m)$ has received a proposal. Correspondingly, w gives m the highest priority. This guarantees that agents not single in period 1 will be weakly better off in Stage 2. Secondly, women in D^W can be regarded as unmatched in μ^I and will become increasingly

with engagements.

weakly better off as the Stage 2 algorithm proceeds since their priority ranking coincides with the preference and they will only reject offers in favor of a better choice (a more preferred proposal or being unmatched). By contrast, $w \in D^W$ may become worse off when her period-1 partner, after being triggered, proposes to w again. Thirdly, note that μ^I is not period-1 blocked by any pair involving someone in $D^P \cup T_3$ and all other kinds of period-1 blocking can be regarded as some form of period-2 blocking, thus the key of the proof is to show that the triggered deferred acceptance algorithm will actually drive away all blocking sets in period 2.

4.1.2 Restriction on Period-2 Blocking: Binding Engagement

The second way to deal with the nonexistence of DSTC matching is to impose more restrictions on period-2 blocking coalitions. Note that in the previous setting, engagements or partnership plans like $\mu(m) = mw$ formed by m and w in period 1, are not binding. In other words, m or w is free to cancel the engagement unilaterally in period 2 when a more preferred candidate is present. This creates a gap between marriage (immediate employment) and engagements (unraveling contracts) since those matched in period 1 can only be weakly better off when there exist breakups and rematches in period 2, while those involved in engagements may be strictly worse off. This contributes to the nonexistence result.

However, in some real-world applications, unravelling is more restrictive. For instance, recall the problem faced by the Association of American Medical Colleges (AAMC) in the early 1940s, where the agreements for internships were typically determined about one or two years before the graduation of medical students. Hospitals were willing to set the date at which they would finalize binding agreements with interns a little earlier than their principal competitors, just because such contracts were binding so that they did not need to worry that those interns may unilaterally break off the engagement and match with other hospitals. Similar examples are also prevalent in other labor markets where the potential employees are restricted by certain constraints (like schooling or minimum age) and thus cannot be available in the first period. More generally, contracts signed before the actual starting time can be regarded as a case with binding engagements to some extent.

By comparison, the situation is more ambiguous in marriage market as engagement typically has no restrictive power in the present legal system. However, in the Mid age or the feudal period of China, such a engagement is a necessary and binding procedure before marriage.

Now we present the formal definition of the above idea about binding engagements.
Def 9. (Period-2 Blocking by a Coalition with Binding Engagements) *$S \in M \cup W$ can period-2 block a matching μ with two-sided commitment and binding engagements if*

i *Mutual involvement, binding engagements:* $\forall x \in S$, $\mu_1(x) \in S$ *and* $\mu_2(x) \in S$;

ii *Implementation and mutual benefits:* $\exists\, \overline{\mu}_2 : S \to S$, *s,t* $\forall x \in S$, $\overline{\mu}_2(x) \in S$ *and* $(\mu_1, \overline{\mu}_2) \succ_S \mu$.

Note that in the above definition, m being engaged with w does not necessarily mean that m should stay single in period 1, that is, m can still be matched with a different woman w' in period 1 while being engaged with w for the future marriage in period 2.

Def 10. (Dynamic Stability with Two-sided Commitment and Binding Engagements, DSTCBE) *In a two-period marriage market* $(M,W,(T_1,T_2,T_3),\overline{R})$, *a matching* μ *is* ***dynamically stable with two-sided commitment and binding engagements*** *if it is not blocked by any individual, not period-1 blocked by any pair of agents, and not period-2 blocked by any coalition with binding engagements.*

Actually, the restriction of binding engagements can guarantee the existence of stability.

Theorem 8. The set of dynamically stable matchings with two-sided commitment and binding engagements is nonempty for any marriage market $(M,W,(T_1,T_2,T_3),\overline{R})$.

A new algorithm based on a combination of triggered DA algorithm and TTC algorithm is put forward to prove the existence result.

Algorithm 3 (Three-Stage TDA) The man-proposing **three-stage triggered deferred acceptance algorithm** identifies a matching μ as follows:

- *Stage 1. Restricted P-DA:*
  Consider a two-period marriage market $(M,W,(T_1,T_2,T_3),\overline{R})$. Define $\overline{R}_1$ as the preferences restricted on certain plans, that is, for agent $m \in M$, $\overline{R}_1(m)$ agrees with $\overline{R}(m)$ on $X_m^1 \equiv \{ww, mm : w \in W\} \cup Y_m^1 \equiv \{wm : w \in W \text{ and } mw \succ_w mm\} \cup Z_m^2 \equiv \{mw : w \in W\}$ and all other plans are regarded as unacceptable.
  Similarly we can define $\overline{R}_1(w)$. Then run the P-DA, which is modified from P-DAFC with $X_m^1 \cup Y_m^1 \cup Z_m^2$ replacing X_m^0 as potential proposed plans for the market $(M,W,\overline{R}_1)$ and get the interim matching μ^I.[12]

- *Stage 2. M-proposing Triggered DA for Period 2 with Binding Engagement:*
  For notations, denote $D^M = \{m : \mu^I(m) = mm\} - T_3$, $D^W = \{w : \mu^I(w) = ww\} - T_3$, $D^{P1} = \{i : \mu_1^I(i) \neq i\} - T_3$, $D^{P2} = \{i : \mu_1^I(i) = i, \mu_2^I(i) \neq i\} - T_3$. Here we exclude type-3 agents

[12] By comparison, in P-DAFC, the proposing side can only propose plans that satisfy full commitment (consistency); here, m can also choose any plan in the form of wm.

since they will definitely be unmatched in period 2 and their partnerships has been fully determined by μ^I. Denote $E^m = \{w \in W : (\mu_1^I(m), w) \succsim_m \mu^I(m) \textit{ and } (\mu_1^I(w), m) \succsim_w \mu^I(w)\}$ as the set of women who can mutually benefit from being matched with m in period 2 with respect to μ^I. Moreover, given the period-2 spot preferences $\succ_w^{2(\mu_1^I)}$, agent w's priority is determined as follows: for $w \in D^W$, the priority is the same as $\succ_w^{2(\mu_1^I)}$; for $w \in D^{P1}$, $\mu_1^I(w)$ has the highest priority and the priority for others coincides with $\succ_w^{2(\mu_1^I)}$; for $w \in D^{P2}$, $\mu_2^I(w)$ has the highest priority and the priority for others coincides with $\succ_w^{2(\mu_1^I)}$.

- *Round 1*: Each man $m_i^2 \in D^M$ proposes to his most preferred agent in $E^{m_i^2}$ (if any);
  If some $w_i^1 \in D^{P1}$ receives her first proposal, then her period-1 partner $\mu_1^I(w_i^1) = m_i^1$ is 'triggered' and will propose to his most preferred agent in $E^{m_i^1}$ (if any);
  If some $w_i^3 \in D^{P2}$ receives her first proposal, then her period-2 partner $\mu_2^I(w_i^3) = m_i^3$ is 'triggered' and will propose to his most preferred agent in $E^{m_i^3}$ (if any);
  Each w holds the acceptable proposal with the highest priority (if any) and rejects all others. By definition of E^m, we know that w can only receive proposals at least as good as $\mu_2^I(w)$.
  $\cdots$ $\cdots$
- *Round k ($\geqslant 2$)*: Each man m rejected in the $(k-1)th$ round proposes to his most preferred agent in $E^m \cap \{w : w \text{ has not rejected } m\}$ (if any);
  If some $w_i^1 \in D^{P1}$ receives her first proposal, then her period-1 partner $\mu_1^I(w_i^1) = m_i^1$ is 'triggered' and will propose to his most preferred agent in $E^{m_i^1}$ (if any);
  If some $w_i^3 \in D^{P2}$ receives her first proposal, then her period-2 partner $\mu_2^I(w_i^3) = m_i^3$ is 'triggered' and will propose to his most preferred agent in $E^{m_i^3}$ (if any);
  Each w holds the acceptable proposal with the highest priority among the new offers and the one she held in the last round(if any), and rejects all others.
  $\cdots$ $\cdots$
- *Outcome*: The algorithm terminates when no more rejections happen. μ_2^{II} matches each m to w who is holding his offer (if any) and for all others, $\mu_2^{II}(i) = \mu_2^I(i)$. $\mu^{II} = (\mu_1^I, \mu_2^{II})$.

- *Stage 3. Restricted Top Trading Cycles:*
  We further define a state variable for those agents in $D^{P1} \cup D^{P2}$:
  $m \in D^{P1} \cup D^{P2}$ is 'untriggered' if $\mu_2^I(m)$ has received no proposal. Denote the set as F^{un};
  $m \in D^{P1} \cup D^{P2}$ is "on" if $\mu_2^I(m)$ has received some proposal and $\mu_2^{II}(m) \neq \mu_2^I(m)$. Denote the set as F^{on};
  $m \in D^{P1} \cup D^{P2}$ is "off" if $\mu_2^I(m)$ has received some proposal and $\mu_2^{II}(m) = \mu_2^I(m)$. Denote

the set as F^{off};

Moreover, $\mu_2^I(m)$ shares the same state as m. We carry out the TCC algorithm, respectively restricted to F^{un} and F^{off}. Specifically, each man points to his most preferred individual in $E^m \cap F^{un}$ ($E^m \cap F^{off}$) and each woman w points to her period-2 partner $\mu_2^I(w)$.[13] TTC mechanism is contributed to David Gale, which is firstly introduced by Shapley and Scarf (1974)[22]. We take the case for F^{un} as an example.

- *Round 1*: Each man $m \in F^{un}$ points to his most preferred agent in $E^m \cap F^{un}$ (if any);
  Each woman $w \in F^{un}$ points to her period-2 partner $\mu_2^I(w)$;
  Carry out all cycles with μ_2^{III} by assigning each m to the w he is pointing to (if any), and remove them from the market. Denote the set of agents that have been removed as G_1.
  … …
- *Round k ($\geqslant 2$)*: Each remaining man $m \in F^{un} - G_{k-1}$ points to his most preferred agent in $E^m \cap (F^{un} - G_{k-1})$ (if any);
  Each remaining man $w \in F^{un} - G_{k-1}$ points to his most preferred agent in $E^w \cap (F^{un} - G_{k-1})$ (if any);;
  Carry out all cycles with μ_2^{III} by assigning each m to the w he is pointing to (if any), and remove them from the market. Denote the set of agents that have been removed (including those removed in previous rounds) as G_k.
  … …
- *Outcome*: The algorithm terminates when no cycle occurs.

After the TTC algorithm is carried out for F^{un} and F^{off} respectively, define that μ_2^{III} coincides with μ^{II} for $(M \cup W) - (F^{un} \cup F^{off})$. $\mu = (\mu_1^I, \mu_2^{III})$ is the ultimate outcome of the man-proposing **three-stage triggered deferred acceptance algorithm**.

Again, there are several important observations. To begin with, $\forall i \in M \cup W$, $\mu(i) \succsim_i \mu^{II}(i) \succsim_i \mu^I(i)$. To see why this is true, note that in Stage 2, for all those matched in μ^I, breakups will happen only when both sides of the partnership are strictly better off. As for those unmatched in Stage 1, they will only get matched in Stage 2 with acceptable partners. Thus, $\forall i \in M \cup W$, $\mu(i) \succ_i \mu^{II}(i)$. Moreover, only agents in $F^{un} \cup F^{off}$ may change their matching in Stage 3 and again, everyone is weakly better off since each $m \in F^{un} \cup F^{off}$ has the outside option of pointing to $\mu_2^I(m)$ maintain the assigned plan in μ^{II}. This guarantees $\forall i \in M \cup W$, $\mu^{II}(i) \succ_i \mu^I(i)$. Secondly, we need a new Stage of restricted TTC mechanisms compared to Algorithm 2, which is used for WDSTC matchings. The key intuition is that in the setting of WDSTC, period-2 blocking coalitions of

[13] Actually, for $i \in F^{un} \cup F^{off}$, $\mu^I(i) = \mu^{II}(i)$.

μ^I cannot occur restricted to the set of agents with commitment (that is, D^P); by contrast, with binding engagements, period-2 blocking coalitions of μ^I exist among agents with commitment (that is, $D^{P1} \cup D^{P2}$). Accordingly, those 'untriggered' or 'off' agents may be mutually beneficial when forming a cycle of partner exchanges. Note that all such agents are matched in period 2, the classcial TTC algorithm can be applied.

4.1.3 Restriction on Preferences: Preference for Early Filling

Thirdly, we would like to keep the original definition of DSTC while imposing more assumptions on the preferences. Recall that the key to the counterexample 4.1 is that $m_3w_2 \succ_{m_3} w_3w_3$, that is, compared to being matched with a mediocre agent for two period, agent m_3 prefers to wait a period for a better agent w_2 to arrive at the market, which implies that w_2 is much better than w_3 and / or m_3 is patient enough. However, such a situation may be ruled out in some real-world applications.

Firstly, the cost of being unmatched in centain period can be so unbearable that the individuals' preferences may exhibit a bias against 'empty seats'. For instance, a supplier that has signed a long-run contract with a retailer cannot stop production in any period and thus desires to hire a mediocre worker to fulfil the minimum production amount required by the contract. Actually, partnerships in different periods may be mutually complementary, instead of being addictive in utility.

Secondly, there may exist some governmental regulations imposed by the social planner to avoid such unravelling phenomena. Dimakopoulos and Heller (2014) study the allocation of German lawyers to different regional courts for their compulsory legal traineeship where some lawyers need to wait for a position. They introduce a criterion called 'early filling', which means policy-makers would not be willing to allow some place at a court to go unfilled simply to allow a current applicant to obtain an allocation to a preferred court in a later period.

To summarize the main idea in the above observations, we put forward an example of behavioral assumptions on preferences.

Def 11. Strong Impatience (SI)
For $m \in T_1$, if $w_1 \in T_1$, $w_2 \in T_2$ and $w_1w_1 \succ_m mm$, then $w_1w_1 \succ_m mw_2$.
For $w \in T_1$, if $m_1 \in T_1$, $m_2 \in T_2$ and $m_1m_1 \succ_w ww$, then $m_1m_1 \succ_w wm_2$.

Theorem 9. For a two-period market $(M, W, (T_1, T_2, T_3), \overline{R})$ where agents in T_1 satisfy SI, the set of dynamically stable matchings with two-sided commitment is nonempty.

With the restriction on preferences, we can use a combination of Stage-1 in Three-stage TDA and Stage-2 in Two-stage TDA algorithm to produce a DSTC matching. For clarity, it is states as below.

Algorithm 4 (Refined Two-Stage TDA) The man-proposing **Refined two-stage triggered deferred acceptance algorithm** identifies a matching μ as follows:

- *Stage 1. Restricted P-DA:*
  Consider a two-period marriage market $(M, W, (T_1, T_2, T_3), \overline{R})$. Define $\overline{R}_1$ as the preferences restricted on certain plans, that is, for agent $m \in M$, $\overline{R}_1(m)$ agrees with $\overline{R}(m)$ on $X_m^1 \equiv \{ww, mm : w \in W\} \cup Y_m^1 \equiv \{wm : w \in W \text{ and } mw \succ_w mm\} \cup Z_m^2 \equiv \{mw : w \in W\}$ and all other plans are regarded as unacceptable.
  Similarly we can define $\overline{R}_1(w)$. Then run the P-DA, which is modified from P-DAFC with $X_m^1 \cup Y_m^1 \cup Z_m^2$ replacing X_m^0 as potential proposed plans for the market $(M, W, \overline{R}_1)$ and get the interim matching μ^I.[14]

- *Stage 2. M-proposing Triggered DA for Period 2:*
  For notations, denote $D^M = \{m : \mu_1^I(m) = m\} - T_3$, $D^W = \{w : \mu_1^I(w) = w\} - T_3$, $D^P = \{i : \mu_1^I(i) \neq i\} - T_3$. Here we exclude type-3 agents since they will definitely be unmatched in period 2 and their partnerships has been fully determined by μ^I. Denote $E^m = \{w \in D^P \cup D^W : (\mu_1^I(m), w) \succsim_m (\mu_1^I(m), \mu_1^I(m))\ \textit{and}\ (\mu_1^I(w), m) \succsim_w (\mu_1^I(w), \mu_1^I(w))\}$ as the set of women who can mutually benefit from matching with w in period 2 with respect to μ^I. Moreover, given the period-2 spot preferences $\succ_w^{2(\mu_1^I)}$, agent w's priority is determined as follows: for $w \in D^W$, the priority is the same as $\succ_w^{2(\mu_1^I)}$; for $w \in D^P$, $\mu_1^I(w)$ has the highest priority and the priority for others coincides with $\succ_w^{2(\mu_1^I)}$.

  - *Round 1*: Each man $m_i^2 \in D^M$ proposes to his most preferred agent in $E^{m_i^2}$ (if any);
    If some $w_i^1 \in D^P$ receives a proposal, then her period-1 partner $\mu_1^I(w_i^1) = m_i^1$ is 'triggered' and will propose to his most preferred agent in $E^{m_i^1}$ (if any);
    Each $w \in D^W \cup D^P$ holds the acceptable proposal with the highest priority (if any) and rejects all others.
    ... ...
  - *Round k ($\geqslant 2$)*: Each man m rejected in the $(k-1)th$ round proposes to his most preferred agent in $E^m \cap \{w : w \text{ has not rejected } m\}$ (if any);
    If some $w_i^1 \in D^P$ who has not been proposed to in the previous rounds receives a proposal, then her period-1 partner $\mu_1^I(w_i^1) = m_i^1$ is 'triggered' and will propose to his most

[14] By comparison, in P-DAFC, the proposing side can only propose plans that satisfy full commitment (consistency); here, m can also choose any plan in the form of wm.

preferred agent in $E^{m_i^1}$ (if any);
Each $w \in D^W \cup D^P$ holds the acceptable proposal with the highest priority among the new offers and the one she held in the last round(if any), and rejects all others.

… …

- *Outcome*: The algorithm terminates when no further rejections happen. μ_2^{II} matches each m to w who is holding his offer (if any) and for all others, $\mu_2^{II}(i) = \mu_2^{I}(i)$.

$\mu = (\mu_1^I, \mu_2^{II})$ is the ultimate outcome of the man-proposing **refined two-stage triggered deferred acceptance algorithm**.

Intuitively, with SI, type-1 agents matched to a consistent plan are unwilling to form a blocking pair with type-2 agents, which resembles the limited blocking power of entries where such a blocking pair is not valid, and thus the existence of DSTC matchings is maintained. For more details, please refer to the proof in the Appendix.

4.2 Properties of Dynamic Stability with Two-sided Commmitment

To study the properties of DSTC matchings, we must firstly guarantee existence. In this section, we follow the third approach of the previous section by imposing assumptions on preferences. Unfortunately, even with bias against 'empty seats', the interactions between spot matchings in different periods make many properties, such as lattice structure and Pareto-efficiency, which are valid in the framework with full commitment, fail to hold with mutual consent dicorce, even if the existence of stability is guaranteed.

Fact 1. Negative Results about Properties of DSTC
In a two-period marriage market $(M, W, (T_1, T_2, T_3), \overline{R})$ where agents in T_1 satisfy SI:

(i). A dynamically stable matching with two-sided commitment can be Pareto-dominated by another dynamically stable matching.

(ii). The refined two-stage TDA algorithm may be not strategy proof for the proposing side.

(iii). There may not exist a M-optimal dynamically stable matching with two-sided commitment.

(iv). The set of dynamically stable matchings with two-sided commitment may not form a lattice under $\succ_M$ or $\succ_W$.

For (i) , consider the following counterexample.
Example 4.2 $M = \{m_1\} = T_1, W = \{w_1, w_2\}, T_2 = \{w_2\}, T_3 = \{w_1\}$ and the preferences:

$m_1 : w_2w_2 \succ_{m_1} w_1w_2 \succ_{m_1} w_2w_1 \succ_{m_1} w_1w_1 \succ_{m_1} m_1w_2 \succ_{m_1} m_1w_1 \succ_{m_1} w_2m_1 \succ_{m_1} w_1m_1 \succ_{m_1} m_1m_1$;
$w_1 : m_1w_1 \succ_{w_1} w_1w_1$; $w_2 : w_2m_1 \succ_{w_2} w_2w_2$

Via M-proposing or W-proposing refined two-stage TDA algorithm, we get the same DSTC matching μ: $\mu(m_1) = m_1w_2$, $\mu(w_1) = w_1w_1$. However there exists another DSTC matching μ': $\mu'(m_1) = w_1w_2$ and $\mu'(m_1) \succ_{m_1} \mu(m_1), \mu'(w_1) \succ_{w_1} \mu(w_1), \mu'(w_2) = \mu(w_2)$. Then μ is Pareto dominated by μ'. The key reason why μ is not blocked by (m_1, w_1) is that when trying to change the partnership in period 1, agents cannot take the period-2 matching as given.

For (ii), consider Eaxmple 4.3 as follows:
Example 4.3 $T_1 = \{m_1, m_2, w_1, w_2\}, T_2 = \{m_3, w_3\}$ and the preferences:
$\mathbf{m_1} : w_3w_3 \quad w_1w_3 \quad w_2w_3 \quad w_1w_1 \quad w_2w_2 \quad m_1m_1$; $\mathbf{m_2} : w_1w_1 \quad w_2w_2 \quad m_2m_2$
$\mathbf{m_3} : m_3w_2 \quad m_3m_3$
$\mathbf{w_1} : m_1m_1 \quad m_2m_2 \quad w_1w_1$; $\mathbf{w_2} : m_3m_3 \quad m_1m_3 \quad m_1m_1 \quad m_2m_2 \quad w_2w_2$
$\mathbf{w_3} : w_3m_1 \quad w_3w_3$

With everyone truthfully reporting his or her preference, the outcome of the M-proposing refined two-stage TDA algorithm is $\mu^1(m_1) = w_1w_1$, $\mu^1(m_2) = w_2w_2$, $\mu^1(m_3) = m_3m_3$ and $\mu^1(w^3) = w_3w_3$. However, if m misreports that he prefers w_2w_2 to w_1w_1, then M-proposing refined two-stage TDA algorithm will produce $\mu^2(m_1) = w_2w_3$, $\mu^2(m_2) = w_1w_1$, $\mu^2(m_3) = m_3w_2$. Note that $\mu^2(m_1) = w_2w_3 \succ_{m_1} w_1w_1 = \mu^1(m_1)$, which means that m_1 is better off by not telling the truth. Thus, M-proposing refined two-stage TDA algorithm is not strategy-proof for men.

As for a counterexample in (iii), we refer to the Example 3 in KK(2015b). The key intuition is that the possibility of rematch may bring about conflicts within the same side of the market. For (iv), if the set of DSFC matchings forms a lattice, we can always produce a M-optimal DSTC matching by repeatedly using $\vee_M$, which contradicts with (iv).

The key intuition behind the above negative results is that, with the possibility of divorce, there may exist mutually profitable cycles invloving both periods comparable to the current DSTC matching, which is not excluded by the definition of DSTC.

The first observation is that without arrivals or departures, the satisfactory properties remain valid. If $T_2 \cup T_3 = \emptyset$, then everyone has rankable preferences, which helps prevent Pareto-improving cycles of assignments in a DSTC matching based on multi-period coordination and rescheduling and thus guarantee efficiency. Moreover, if agents can only report rankable preferenecs, then the

M-proposing T-DA algorithm is streategy-proof for all men. Actually, under rankabality, refined two-stage TDA algorithm degenerates to P-DAFC algorithm and thus we get DATC matching with a much simpler procedure.

Theorem 10. (Efficiency and Restricted Strategy-proofness) In a two-period matching market $(M,W,(T_1,T_2,T_3),\overline{R})$ where $T_2 \cup T_3 = \emptyset$,
(1). All dynamically stable matching with two-sided commitment is Pareto efficient within the set of individually rational matchings.
(2). If all agents can only report preferences that satisfy rankability, then the T-DA is strategy-proof for the proposing side.

Here we want to study arrivals and departures. To reconcile the conflicts shown in Example 4.2, we can restrict the patterns of variation in the underlying set of agents. Note that in Example 4.2, there exist both entries and exits on the same side of the market. If this condition does not hold, a DATC matching is not dominated by any individually rational matching.

Proposition 1. For a two-period matching market $(M,W,(T_1,T_2,T_3),\overline{R})$ where SI holds,

(1) If either $(T_2 \cap W) \cup (T_3 \cap M) = \emptyset$ or $(T_2 \cap M) \cup (T_3 \cap W) = \emptyset$ or $(T_2 \cap M) \cup (T_2 \cap W) = \emptyset$, then all dynamically stable matching with two-sided commitment is Pareto efficient within the set of individually rational matchings.

(2) If $(T_2 \cap W) \cup (T_3 \cap M) \neq \emptyset$ and $(T_2 \cap M) \cup (T_3 \cap W) \neq \emptyset$, and $(T_2 \cap M) \cup (T_2 \cap W) \neq \emptyset$, then there exists some market $(M,W,(T_1,T_2,T_3),\overline{R})$ such that the outcome DSTC matching of refined two-stage TDA algorithm is pareto dominated by another DSTC matching.

The first part in Proposition 1 shows that when there are no arrivals and departures on the same side, or arrivals on both sides, any DSTC matching is not Pareto dominated by any individually rational matching. The second part further implies the efficiency result cannot hold if such conditions fail. Thus, to incorporate more general patterns of variations of agents, more assumptions on prefernces are required to rule out the counterexamples in Proposition 1.

Assumption (A1): Strong Cost of Separation
(1). For $m_1 \in T_1$, $w_1 \in T_3$, $w_2 \in T_2$. If $w_1w_2 \succ_{m_1} w_3w_3$ for some $w_3 \neq w_1 \in W$, then $w_1m_1 \succsim_{m_1} w_3w_3$.
(2). For $m_1, w_1 \in T_1$, $w_2 \in T_2$. If $w_1w_2 \succ_{m_1} w_3w_3$ for some $w_3 \neq w_1 \in W$, then $w_1w_1 \succsim_{m_1} w_3w_3$.

$A1$ means that the cost of divorce is rather high and thus when some inconsistent plan is preferred to another consistent partnership, its first-period 'component' should be good enough. Here by 'component' we mean the reasonable partnership given the type of the agent, that is, w_1w_1 if

$w_1 \in T_1$ and $w_1 m_1$ if $w_1 \in T_3$. Note we do not require that the component is preferred to the original inconsistent plan $w_1 w_2$. It is true that such an assumption is pretty restrictive, but in order to allow for any kind of arrivals and departures and rule out the counterexamples in Proposition 1, it is better to put forward a group of sufficient conditions.

Proposition 2. For a two-period matching market $(M, W, (T_1, T_2, T_3), \overline{R})$ where SI and Strong Cost of Separation holds, if μ is the outcome of refined two-stage TDA algorithm, then μ is Pareto efficient within the set of individually rational matchings.

The above theorem shows that with certain assumptions, all DSTC matchings as the outcome of refined two-stage TDA algorithm can be on the efficient frontier of individually rational matchings, no matter how the underlying set of agents change over time.

One possible extension is to include limited commitment and monetary transfers. For instance, in the marriage market, engagement can be disobeyed unilaterally with some amount of transfer. In the commercial contracting market, a fixed amount of liquidated damages is usually specified in the terms. It is possible that such kind of monetary transfers can be so overwhelming that one would rather carry out the contract as a result of bargaining in the negotiation process. We can also argue that by the Coase Theorem, interior transfers will always produce the mutually efficient result. If the initial right lies in the persistent side as suggested by the marriage vows, divorce can only happen when the transfer can make both sides better off, which can be regarded as agreed divorce after the payment has been incorporated into the preferences. **Nevertheless, it is not that convincing unless we explicitly discuss and justify the transfers and costs** and the present model with two-sided commitment just serves as an extreme contrast to the classical model without commitment, which provides insights about the difference before going into cardinal models.

5 Matching with One-sided Commitment

In the real-world example of two-sided markets, the two sides are typically not treated equally. Actually, the market structure is usually designed, by nature or by the market maker, in favour of one side, based on a combination of different factors like fairnes, efficiency, relative bargaining power and the utility function of the social planner.

Many markets where humans are matched to positions or services exhibit such a partiality for humans. Take the labor market for an example. On the condition that both sides behave themselves according to the terms and clauses, the labor contract remains valid until its expiration without any intervention. That is, agents would like to divert from current matching only if they find a better achievable choice. As is indicated by the corresponding law clauses, employees have the right to resign from the current job given that they inform the employers in advance. By contrast, however, employers cannot freely fire the workers simply in favour of another applicant. To concentrate on the influence of preferences solely, we ignore the possibilities of the cancellation of contracts because of misbehaviour on either side, that is, all breakups are no-fault. Then only the workers can freely break the contract while the employers can only provide a position for another preferred agent when the current worker voluntarily quits. For concrete instances, one may think of the job market for tenured Economics professors.

Although currently men and women are perceived to be equal both in terms of legal terms and social norms, similar situations were also present in the marrigae market of feudal societies such as ancient China, where gender discrimination was prevalent against women. Chastity was then considered to be a treasured virtue of women, who were expected to be loyal to their husbands. By contrary, men could write a certificate of divorce to break a relationship unilaterally as they preferred. This accords with our following definition of one-sided commitment.

Such asymmetric market structure has been studied in some specific settings in the literature. As an example of the labor market, Pereyra (2013)[15] studies the problem of assigning positions to overlapping generations of teachers faced by the Mexican Ministry of Public Education, where individual rationality requires that from one period to another, teachers can either retain their current positions or choose a preferred one. Similarly, Kennes, Monte and Tumennasan (2014a, b)[11][12] examine the dynamic centralized allocation of children to public daycares in Denmark, where school-age children arrive at and leave the market in the pattern of overlapping generations while institutions are persistent in the market. Although one-sided commitment is not explicitly expressed, each institution gives the highest priority to its currently enrolled children, which implicitly means that the children can unilaterally maintain the period-1 matching. Another instance is the dynamic dormitory allocation problem in the college studied by Kurino (2014)[14]. Typically, the Department of Housing will allow students the opportunity to retain their previously occupied rooms to guarantee the satisfaction of customers and prevent them from living off campus.

5.1 Model setup and Notion of Stability

We now change to the language of multi-period school choice problems for ease of illustration. S_t (C_t), $t = 1,2$ defines the set of students (colleges) available to be matched. Up till now, we stay with the simple case of one-to-one matching market by assuming that the capacity of each college

$q_c = 1$. Denote $C^s \equiv C \cup \{s\}$ and $S^c \equiv S \cup \{c\}$. Any agent $s \in S_1 \cap S_2$ ($c \in C_1 \cap C_2$) holds a strict and rational preference $\succ_s^0$ ($\succ_c^0$) over partnership plans $(x_1, x_2) \in C_1^s \times C_2^s$ ($(x_1, x_2) \in S_1^c \times S_2^c$), or $x_1 x_2$ for abbreviation when confusion is unlikely. Similarly, agents only appearing on the market for one period will hold spot strict and rational preferences as those on a static market. Then $\mu = (\mu_1, \mu_2) : S_1 \cup S_2 \cup C_1 \cup C_2 \to (S_1 \cup C_1) \times (S_2 \cup C_2)$ is a **multi-period matching** if $\mu_t : S_t \cup C_t \to S_t \cup C_t$ is a spot matching on the static market for period t. Again, for convenience of notation, we refer to μ as a matching. Then $(S_1 \cup S_2, C_1 \cup C_2, R)$ is a two-period matching market. Similarly, we typically talk about the extended school choice problem $(S, C, \overline{R})$ where $S = S_1 \cup S_2$ and $C = C_1 \cup C_2$.

To define individual rationality, one should be careful about the asymmetry between two sides. Notice that students cannot only force the current schooling to persist in the second period, but can also drop out or transfer to another school freely, which is the combination of corresponding notions in KK (2015a) and the market with two-sided commitment. Specifically, agent $i \in C \cup S$ can **period-1 block** a matching μ if $ii \succ_i \mu(i)$; agent $s \in S$ can **period-2 block** μ if either $(\mu_1(s), \mu_1(s)) \succ_s \mu(s)$ or $(\mu_1(s), s) \succ_s \mu(s)$. μ is **individually rational** if it is not period-1 or period-2 blocked by any individual.

A matching μ is **period-1 blocked by a pair of agents** (s, c) if either (i) $ss \succ_c \mu(c)$ and $cc \succ_s \mu(s)$ or (ii). $cs \succ_c \mu(c)$ and $sc \succ_s \mu(s)$ or (iii). $sc \succ_c \mu(c)$ and $cs \succ_s \mu(s)$. [15]

For coalitional blocking in period 2, we can follow the case with two-sided commitment to put forward different ways to guarantee the existence of stability in a market with arrivals and departures. However, to avoid similar discussions, we will take the model with binding engagements as an example in this section, which may suit real-world applications relatively better given the prevalence of unravelling phenomena such as early admissions in the college admissions problem and pre-signed labor contracts between prospective graduates and employers. Moreover, in contrast with the case with two-sided commitment, binding engagements only require that colleges choosing to be unmatched in period 1 should be committed to period-2 partners.

[15] Actually, in the definition of period-1 pairwise blocking, we can also add the extra conditions on s, which means that the blocking matching should not blocked by either component of the pair, by replacing (i) with (i') that $ss \succ_c \mu(c)$, $cc \succ_s \mu(s)$ and $cc \succ_s cs$ and replacing (iii) to (iii') that $sc \succ_c \mu(c)$, $cs \succ_s \mu(s)$ and $cs \succ_s cc$. Intuitively, with rational expectations that students can unilaterally terminate a partnership or force it to continue in period 2, college c knows that when forming a blocking pair with s, s will definitely manage to be matched with the better plan between cs and cc. Thus, if $cs \succ_s cc$, then any blocking pair involving (c, s) should assign s to $\mu'(s) = cs$; by comparison, if $cc \succ_s cs$, then any blocking pair involving (c, s) should assign s to $\mu'(s) = cc$. In this way, we do not need the assumption of weak rankability w.r.t empty position in most of the results.

Def 12. (Period-2 Blocking by a Coalition with Binding Engagements) *$A \in S \cup C$ can period-2 block a matching μ with one-sided commitment and binding engagements if*

i Asymmetric involvement: $\forall c \in A \cap C$, $\mu_1(c) \in A$ and if $\mu_1(c) = c$, then $\mu_2(c) \in A$;

ii Implementation and mutual benefits: $\exists\, \overline{\mu}_2 : A \to A$, s,t $\forall x \in A$, $\overline{\mu}_2(x) \in A$ and $(\mu_1, \overline{\mu}_2) \succ_A \mu$.

Def 13. (Dynamic Stability with One-sided Commitment and Binding Engagements, DSOCBE) *In a two-period matching market $(S, C, \overline{R})$, a matching μ is **dynamically stable with one-sided commitment and binding engagements** if it is not blocked by any individual, not period-1 blocked by any pair of agents and not period-2 blocked by any coalition.*

5.2 Existence and Properties of Stability: Without Arrivals of Students

With the general group of preferences, the existence of DSOCBE may fail as is suggested in the following example.

Example 5.1 Consider $S = \{s\}$, $C = \{c_1, c_2\}$ and the preferences are as follows:

$s : c_2c_2 \succ_s c_1c_2 \succ_s c_1c_1 \succ_s ss$; $\quad c_1 : ss \succ_{c_1} c_1c_1$; $\quad c_2 : c_2s \succ_{c_2} c_2c_2$

There are three possible cases with acceptable plans for s. (i). If $\mu(s) = c_1c_2$, then $\mu_1(c_1) = s$ but $\mu(c_1) \neq ss$, thus μ is period-1 blocked by the agent c_1 since $\mu(c_1)$ is unacceptable for c_1. (ii). If $\mu(s) = c_1c_1$, the $\mu(c_2) = c_2c_2$, and thus μ is period-2 blocked by agent $A = \{s, c_2\}$ via $\overline{\mu}_2(s) = c_2$ as $c_1c_2 \succ_s \mu(s) = c_1c_1$ and $c_2s \succ_{c_2} c_2c_2 = \mu(c_2)$. (iii). If $\mu(s) = ss$, then $\mu(c_1) = c_1c_1$ and μ is period-1 blocked by the pair (s, c_1) with $\mu'(s) = c_1c_1$ as $c_1c_1 \succ_s \mu(s) = ss$ and $ss \succ_{c_1} \mu(c_1) = c_1c_1$. Thus no DSOCBE matching exists in this market.

As in the case with two-sided commitment, if we are considering a market with a fixed set of agents, then rankability will again guarantee nice results for existence and efficiency. However, a set of weaker assumptions on preferences can achieve the same goal. We begin with the conditions used in Kadam and Kotowski (2015a,b)[9][10].

Assumption 5.1 (Sequential improvement complementarity, SIC)[16] $\succ_m$ satisfies Sequential improvement complementarity if (i). $jk \succ_m jj \succsim_m mm \implies kk \succ_m jj$; (ii). $jk \succ_m jm \succ_m mm \implies kk \succ_m jm$; (iii). $mk \succ_m mj \succsim_m mm \implies kk \succ_m mj$.

[16] In a updated version of Kadam and Kotowski (2015a)[9], the new definition of 'SIC' is just SIC-(i) here, and the combination of SIC-(ii) and SIC-(iii) is called revealed dominance of singlehood (RDS). This helps them to clarify the relationships between different assumptions, but will not affect the main results. For simplicity, we still use the concept of 'SIC' as a whole.

Intuitively, by naming it sequential improvement, we mean that if an agent prefers to switch assignments after period 1 ranther than maintaining her initial assignment or being unmatched in period 2, then the change must be made towards a better option. There are a couple of observations worth mentioning. On the one hand, SIC is compatible with departures, but incompatible with arrivals. To see this, if s appears on the market in period 2 and $sc \succ_s ss \succ_s cc$, , then by SIC-(i), $cc \succ_s ss$, which is a contradiction. On the other hand, SIC does not exclude preferences for variations as it is possible that $c_1c_2 \succ_s c_2c_2 \succ_s c_1c_1$, which is ruled out by rankability. Moreover, departures are compatible with SIC, although this does not hold for rankability.

In Kadam and Kotowski (2015a,b)[9], SIC can guarantee the existence of their definition of dynamic stability (or dynamic stability with no commitment in our framework). However, such a condition is not sufficient for the case with one-sided commitment.

Example 5.2 Consider $S = \{s_1, s_2\}$, $C = \{c\}$ and the preferences are as follows:

$$s_1 : cc \succ_{s_1} cs_1 \succ_{s_1} s_1s_1; \quad s_2 : cc \succ_{s_2} s_2s_2; \quad c : s_1c \succ_c s_2s_2 \succ_c cc \succ_c s_1s_1$$

It is clear that the above preferences satisfy SIC. However there is no dynamically stable matching with one-sided commitment in this market. Specifically, there are three cases. (i). If $\mu(s) = cc$, then μ is period-1 blocked by the agent c since it is unacceptable for c. (ii). If $\mu(s) = cs$, the μ is period-2 blocked by agent s since s can individually force the partnership to sustain. (iii). If $\mu(s) = ss$, then $\mu(c) = s's'$ and μ is period-1 blocked by the pair (s,c) with $\mu'(s) = cs$. All cases lead to unstable matchings.

Thus, additional assumptions may be required. Typically, in the asymmetric market, one side consists of institutions and thus it is reasonable that the college will prefer the positions to be filled by acceptable agents. That is, if a plan where a student attends the college in period 1 and drops out in period 2 is acceptable, then the plan where the same student remains in the college in both periods is acceptable as well.

Assumption 5.2 (Weak Rankability w.r.t Empty Positions (WREP)) The preference of college c is said to exhibit weak rankability w.r.t empty positions if $sc \succ_c cc \Longrightarrow ss \succ_c cc$.

Now we can state the existence result for DSOCBE matchings.

Theorem 11. In a two-period matching market $(S, C, \overline{R})$, if everyone's preferences satisfy SIC and $\forall c \in C$, $\succ_c$ exhibits WREP, then the set of dynamically stable matchings with one-sided commitment and binding engagements is non-empty.

The corresponding stable algorithm is adjusted from the PDA-FC algorithm. Firstly, we allow for proposals wihtout full commitment (like proposal cs made by s) in the Stage 1 algorithm; Secondly, we include Stage 2 to allow those unmatched in period 2 to find their desirable period-2 partner in a spot DA algorithm.

Algorithm 5 (PDA-OC)) The student-proposing **plan deferred acceptable algorithm with adjustment** identifies a matching μ^* as follows:

- *Stage 1. P-DA:*
  Consider a two-period college admission problem $(C,S,\overline{R})$. Define $\overline{R}_1$ as the preferences restricted on certain plans, that is, for agent s, $\overline{R}_1(s)$ agrees with $\overline{R}(s)$ on $X_s^4 \equiv \left\{cs, cc, sc, ss : c \in C\right\}$ and all other plans are regarded as unacceptable. Then run the P-DA, which is modified from P-DAFC with X_s^4 replacing X_m^0 as potential proposed plans for the market $(C,S,\overline{R}_1)$ and get the interim matching μ^I.

- *Stage 2. Adjustment for those Unmatched in Period 2*

  - *Step 1*: Define $\overline{R}_2$ as spot preferences conditional on the first-period partnership μ_1^I, that is, $\overline{R}_2(i) = \succ_i^{2(\mu_1^I)}, \forall\, i \in S \cup C$. Denote $C_1 \equiv \{c \in C : \mu_2^I(c) = c\}$, $S_1 \equiv \{s \in S : \mu_2^I(s) = s\}$
  - *Step 2*: Conduct the spot market S-proposing DA algorithm for $(S_1, C_1, \overline{R}_2)$ and get μ_2^{II}.
  - *Step 3*: Define the ultimate outcome as μ^* where $\forall i \in C_1 \cup S_1, \mu^*(i) = (\mu_1^I(i), \mu_2^{II}(i))$ and $\forall i \in (C \backslash C_1) \cup (S \backslash S_1), \mu^*(i) = \mu^I(i)$.

It is easy to see that the result of the Stage 1 algorithm μ^I is not period-1 blocked by any individual or pair due to the property of a variation of the DA algorihm. In stage 2, although only agents unmatched in μ_2^I have the opportunity to rematch in period 2, the final outcome is not period-2 blocked. Intuitively, without arrivals in the second period, if a student s matched to the same college c for two periods in the interim matching μ^I and she desires to switch to another college c', then c' is also open in the first period, and by SIC, student s would prefer to attend college c' for two periods. The same argument holds for c' if the college would like to enroll s compared to $\mu_2(c')$. Thus, if a period-2 blocking coalition involves c' and s, then (s,c') would period-1 block μ with a consistent partnership plan.

Now it is time to examine properties of DSOCBE matchings. We firstly focus on restricted strategy-proofness of the PDA-OC algorithm. By 'restricted', we mean that agents are required to report preferences that satisfy the conditions that are suffcient to guarantee the existence of

DSOCBE matchings with respect to the stated preferences.[17]

Theorem 12. Restricted Strategy-proofness In a two-period matching market $(S, C, \overline{R})$ where everyone's preferences satisfy SIC and $\forall c \in C$, $\succ_c$ exhibits WREP, if all agents can only report preferences that satisfy those assumptions, then the S-proposing PDA-OC is strategy-proof for students. Moreover, it is strategy-proof for any coalition of students.

The intuition coincides with strategy-proofness in a static market or a multi-period market with full commitment, where each one on the proposing side has been matched to their most 'preferred' achievable partnership, measured by the reported preferences in the PDA-OC algorithm and thus the result of truthful reporting will offer them the best plan under the actual preference. By comparison, the key reason for imposing restrictions on reported preferences is to guarantee the feasibility of PDA-OC algorithm under stated preferences.

However, the property of efficiency does not hold so easily, even when we restrict our attention to the set of DSOCBE matchings and rankable preferences. The following example 5.3 shows that the result of PDA-OC mechanism can be Pareto dominated by another DSOCBE matching.
Example 5.3 $S = \{s_1, s_2, s_3, s_4\}, C = \{c_1, c_2, c_3, c_4\}$ and the preferences:

$\mathbf{s_1}: c_2c_2 \quad c_3c_2 \quad c_1c_2 \quad c_3c_3 \quad s_1s_1 \quad c_1c_1;$

$\mathbf{s_2}: c_1c_1 \quad c_4c_1 \quad c_2c_1 \quad c_4c_4 \quad s_2s_2 \quad c_2c_2;$

$\mathbf{s_3}: c_4c_4 \quad c_2c_4 \quad c_3c_4 \quad c_2c_2 \quad s_3s_3 \quad c_3c_3;$

$\mathbf{s_4}: c_3c_3 \quad c_1c_3 \quad c_4c_3 \quad c_1c_1 \quad s_4s_4 \quad c_4c_4;$

$\mathbf{c_1}: s_1s_1 \quad s_1s_4 \quad s_1s_2 \quad s_4s_4 \quad c_1c_1 \quad s_2s_2$

$\mathbf{c_2}: s_2s_2 \quad s_2s_3 \quad s_2s_1 \quad s_3s_3 \quad c_2c_2 \quad s_1s_1$

$\mathbf{c_3}: s_3s_3 \quad s_3s_1 \quad s_3s_4 \quad s_1s_1 \quad c_3c_3 \quad s_4s_4$

$\mathbf{c_4}: s_4s_4 \quad s_4s_2 \quad s_4s_3 \quad s_2s_2 \quad c_4c_4 \quad s_3s_3$

It is simple to check that everyone's preference satisfies SIC, WREP and rankability. In the S-proposing and C-proposing PDA-OC algorithm, the outcome is the same μ where $\mu(s_1) = c_3c_3$, $\mu(s_2) = c_4c_4$, $\mu(s_3) = c_2c_2$ and $\mu(s_4) = c_1c_1$. However, if we consider μ' such that $\mu'(s_1) = c_1c_2$, $\mu'(s_2) = c_2c_1$, $\mu'(s_3) = c_3c_4$ and $\mu'(s_4) = c_4c_3$, then μ' is DSOCBE and everyone is strictly better off in μ' than in μ.

Recall the definiton of two-sided commitment, μ' is not individually rational (and thus not DSTC) in the sense of two-sided commitment since $s_4s_4 \succ_{c_4} s_4s_2 = \mu(c_4)$. That is why μ is efficient among all IR matchings in a market with two-sided commitment, but inefficient in a market with one-sided commitment. Under the asymmetric settings, colleges do not have the ability to enforce the period-1 student to attend in the future, so that they are satisfied with a partnership plan where

[17] A justification for the validity of restricted strategy-proofness is that it is common knowledge (or known by the centralized clearing house) that everyone's preference satisfies SIC and colleges' preferences exhibit WREP.

the second-period partner is relatively worse than the first-period counterpart. Thus, one way to guarantee efficiency is to place more weight to prevent such a situation from happening. That is, when an inconsistent plan involving two different agents is preferred, then the period-2 partner should not be too unsatisfactory.

Assumption 5.3 (Strong Bias towards Final Outcome (SBFO)) The preference of college c is said to exhibits strong bias towards final outcome if $s_1s_2 \succ_c x_1x_2$ for $x_1, x_2 \in S^c \Longrightarrow s_2s_2 \succsim_c x_1x_2$.

Admittedly, such a condition is pretty strong as it requires that either $s_2s_2 \succ_c s_1s_2$ or s_2s_2 is the very plan that is ranked by c behind s_1s_2. To provide a justification, consider that in many situations, institutions care more about the final outcome, that is, the period-2 partner. For instance, a university will place more weight on the performance of graudates instead of those who drop out in the midway. Similarly, agents may have reference-dependent preferences and thus the mental loss incurred by changing to a relatively worse partner in the second period could be great enough to narrow the utility gap between plans s_1s_2 and s_2s_2.

Moreover, given that SBFO holds, SIC is also not sufficient to produce efficiency. See the following example for an illustration.

Example 5.4 $S = \{s\}, C = \{c_1, c_2\}$ and the preferences:

s : $c_2c_1 \quad c_1c_1 \quad c_2c_2 \quad ss$; **c$_1$** : $c_1s \quad ss \quad c_1c_1$ **c$_2$** : $ss \quad sc_2 \quad c_2c_2$

SIC is satisfied for everyone and $\succ_{c_1}, \succ_{c_2}$ exhibits WREP and SBFO. Consider the two versions of PDA-OC algorithm will both produce μ such that $\mu(s) = c_1c_1$, $\mu(c_2) = c_2c_2$. There exists another DSOCBE matching $\mu'(s) = c_2c_1$ such that μ' pareto dominates μ. Thus, we need a stronger version of SIC which modifies the first and the third condition to make the consistent plan kk more preferred. Notice that even under SSIC, departures are possible.

Assumption 5.1' (Strong Sequential improvement complementarity, SSIC) $\succ_m$ satisfies Strong Sequential improvement complementarity if (i). $jk \succ_m jj \Longrightarrow kk \succ_m jk$; (ii). $jk \succ_m jm \succ_m mm \Longrightarrow kk \succ_m jm$; (iii). $mk \succ_m mj \succsim_m mm \Longrightarrow kk \succ_m mk$.

Intuitively, compared to SIC, SSIC more resembles rankability. Now we can have the efficiency result.

Theorem 13. Efficiency In a two-period matching market $(S, C, \overline{R})$ where everyone's preferences satisfy SSIC and $\forall c \in C$, $\succ_c$ exhibits WREP and SBFO, then all DSOCBE matchings are Pareto efficient among the set of individually rational matchings.

5.3 Existence and Properties of Stability: With Arrivals of Students

To guarantee the existence of dynamically stable matchings, Kadam and Kotowski (2015a[9]) assume SIC and point out that the condition is almost necessary in the sense that relaxing any of the three parts of the assumption will induce a counter-example where no stable matching exists. However, if there exists some agent who enters the market in period 2, that is, she strictly prefers to remain unmatched in period 1, the condition is violated. For example, consider s_1 has the preference $s_1c_1 \succ_{s_1} s_1c_2 \succ_{s_1} s_1s_1 \succ_{s_1} c_1c_1 \succ_{s_1} c_2c_2$. With SIC(i), $s_1c_1 \succ_{s_1} s_1s_1$ implies that $c_1c_1 \succ_{s_1} s_1s_1$, a contradiction. With SIC(iii), $s_1c_1 \succ_{s_1} s_1c_2 \succ_{s_1} s_1s_1$ implies that $c_1c_1 \succ_{s_1} s_1c_2$, a contradiction again. Based on this, we can find a market with arrivals of students where no dynamically stable matching exists.

Example 5.5 Consider $S_I = \{s_2\}, S_{II} = \{s_1\}$, $C = \{c_1, c_2\}$ with the following preferences,

$s_1 : s_1c_1 \succ_{s_1} s_1c_2 \succ_{s_1} s_1s_1 \succ_{s_1} c_1c_1 \succ_{s_1} c_2c_2$; $\quad$ $s_2 : c_1c_1 \succ_{s_2} s_2s_2$

$c_1 : s_2s_1 \succ_{c_1} s_1s_1 \succ_{c_1} s_2s_2 \succ_{c_1} c_1s_1 \succ_{c_1} c_1c_1$; $\quad$ $c_2 : s_1s_1 \succ_{c_2} s_2s_2 \succ_{c_2} c_2c_2$

Except for s_1, other agents have standard preferences as before.To guarantee individual rationality, $\mu(c_2) = c_2c_2$.

If $\mu(c_1) = c_1c_1$ or c_1s_1, then μ is period-1 blocked by (s_2, c_1) with $\mu'(c_1) = s_2s_2$.

If $\mu(c_1) = s_2s_2$, then μ is period-2 blocked by (s_1, c_1) with $\mu'(c_1) = s_2s_1$.

If $\mu(c_1) = s_1s_1$, then μ is period-1 blocked by c_1 with $\mu'(c_1) = c_1c_1$

If $\mu(c_1) = s_2s_1$, then μ is period-1 blocked by s_2 with $\mu'(s_2) = s_2s_2$. Thus there is no dynamically stable matching in the market.

However, it can be shown that $\mu^*(c_1) = s_2s_2$, $\mu^*(c_2) = c_2c_2$ and $\mu^*(s_1) = s_1s_1$ is dynamically stable with one-sided commitment since agent s_2 can force the matching $\mu_1^*(s_2) = c_1$ to persist in the second period. Moreover, μ^* can be achieved with PDA-OC. This suggests that the matching market with one-sided commitment may be compatible with arrivals and departures.

Notice that whether arrivals and departures should be regarded as agents with special preferences and ordinary blocking power or simply agents with limited blocking power can influence the market behavior significantly as has been discussed in the case with two-sided commitment. We will examine two ways to incorporate entries of students into the market with one-sided commitment.

5.3.1 Restriction on Period-1 Blocking: Limited Blocking Power of Entries

Firstly, as has been discussed in the case of two-sided commitment, we regard those new-comers as special agents who only exist and affect the period-2 market, or equivalently, they have limited blocking power and can not propose or receive applications in the first period or serve as part of the blocking pairs in period 1. Meanwhile, we keep the assumptions for the other agents. Examples include the situations where there is incomplete information and the period-1 agents cannot ex-ante know the (features of) participants arriving in period 2, or agents cannot sign or enforce an unravelling contract with those who are not currently on this market even with complete information. For a more detailed justification, please refer to section 4.1.1.

Define S_I as the set of students appearing the market in the first period and S_{II} the set of students entering the market in period 2. $S \equiv S_I \cup S_{II}$. Unlike the case with two-sided commitment, we do not specify departures since they do not interfere with standard assumptions of SIC. Now a two-period matching market can be expressed as $(S_I, S_{II}, C, \overline{R})$. Then we need to slightly modify the notions of stability.

Def 14. (Weak Dynamic Stability with One-sided Commitment and Binding Engagaments, WDSOCBE) *In a two-period matching market $(S_I, S_{II}, C, \overline{R})$ with $S \equiv S_I \cup S_{II}$, a matching μ is **weakly dynamically stable with one-sided commitment and binding engagaments** if it is not period-1 blocked by any individual in $C \cup S$ or pair of agents in $C \cup S_I$, and not period-2 blocked by any individual $s \in S$ or any coalition $A \subset S \cup C$.*

Luckily, with limited blocking power, those new-comers in period 2 can be added to the model with one-sided commitment directly while maintaining the existence of the corresponding stable matchings.

Theorem 14. In a two-period matching market with arrivals $(S_I \cup S_{II}, C, \overline{R})$ where everyone $\in S_I \cup C$ has preferences satisfying SIC and $\forall c \in C$, $\succ_c$ exhibits WREP, then the set of weakly dynamically stable matchings with one-sided commitment and binding engagements is nonempty.

A modified version of PDA-OC algorithm will guarantee the existence of WDSOCBE matchings. Actually, the only difference is that we do not allow agents in S_{II} to make proposals in the Stage 1 algorithm here.

Algorithm 6 (PDAAE) The student-proposing **plan deferred acceptable algorithm with adjustment and entry** identifies a matching μ^* as follows:

- *Stage 1. P-DA for $C \cup S_I$:*
  Consider a two-period college admission problem $(C, S, \overline{R})$. Define $\overline{R}_1$ as the preferences restricted on certain plans, that is, for agent s, $\overline{R}_1(s)$ agrees with $\overline{R}(s)$ on $X_s^4 \equiv \{cs, cc, sc, ss : c \in C\}$ and all other plans are regarded as unacceptable. Then run the P-DA, which is

modified from P-DAFC with X_s^4 replacing X_m^0 as potential proposed plans for the market $(C, S_I, \overline{R}_1)$ and get the interim matching μ^I and for $s \in S_{II}$, $\mu^I(s) = ss$.

- *Stage 2. Adjustment for those Unmatched in Period 2*
  - *Step 1*: Define $\overline{R}_2$ as spot preferences conditional on the first-period partnership μ_1^I, that is, $\overline{R}_2(i) = \succ_i^{2(\mu_1^I)}, \forall\, i \in S \cup C$. Denote $C_1 \equiv \{c \in C : \mu_2^I(c) = c\}$, $S_1 \equiv \{s \in S : \mu_2^I(s) = s\}$
  - *Step 2*: Conduct the spot market S-proposing DA algorithm for $(S_1, C_1, \overline{R}_2)$ and get μ_2^{II}.
  - *Step 3*: Define the ultimate outcome as μ^* where $\forall i \in C_1 \cup S_1, \mu^*(i) = (\mu_1^I(i), \mu_2^{II}(i))$ and $\forall i \in (C \backslash C_1) \cup (S \backslash S_1), \mu^*(i) = \mu^I(i)$.

According to the specific form of the PDAAE algorithm, we can directly derive the following intuitive comparative statics concerning the set of arrivals of students in period 2. Specifically, when there are more students arriving at period 2, then the set of colleges will be weakly better off. Note that in a static market, such a property is held primarily because that the outcome of DA algorithm is optimal for the proposing side, which is typically not true in the dynamic market.

Theorem 15. Consider two two-period matching markes with arrivals $(S_I, S_{II}, C, \overline{R})$ and $(S_I, S'_{II}, C, \overline{R})$ with $S_{II} \subset S'_{II}$.Assume that everyone $\in S_I \cup C$ has preferences satisfy SIC and $\forall c \in C$, $\succ_c$ exhibits WREP. Denote μ as the PDAAE outcome for $(S_I, S_{II}, C, \overline{R})$ and μ' as the PDAAE outcome for $(S_I, S'_{II}, C, \overline{R})$, then $\mu' \succsim_C \mu$ and $\mu' \sim \mu$ if and only if $\forall s \in S'_{II} \backslash S_{II}, \mu'(s) = ss$.

5.3.2 Restriction on Preferences

What if we treat these arrivals as agents who appear in the market in period 1 but can only sign an unraveling contract? A good scenario may be the relationship between interns and return offers, or the famous phenomenon of unraveling of NRMP (Roth and Sotomayor, 1990). At around 1940 before there is a centralized clearing house, labor contracts are typically signed as early as 2 years before graduation. That is, although the workers cannot begin working soon after the contracts, their information has been at least partially available for the employers and they also care about their future careers during studying time in the school. With complete information, then even those entering the market in period 2 can participate in the algorithm and blocking process in period 1. In this way, our notions of stability should return to the standard ones.

As we have discussed (see the following Example 5.5 for an example), with arrivals, SIC for everyone and WREP for colleges are no longer enough to guarantee the existence of DSOCBE matchings. Accordingly, more assumptions on preferences are required.

Example 5.6 Consider $S_I = \{s_2\}, S_{II} = \{s_1\}$, $C = \{c_1, c_2\}$ with the following preferences,

$s_1 : s_1c_1 \succ_{s_1} s_1c_2 \succ_{s_1} s_1s_1 \succ_{s_1} c_1c_1 \succ_{s_1} c_2c_2;$ $\quad$ $s_2 : c_2c_2 \succ_{s_2} c_1s_2 \succ_{s_2} s_2s_2$

$c_1 : s_1s_1 \succ_{c_1} s_2s_1 \succ_{c_1} s_2s_2 \succ_{c_1} s_2c_1 \succ_{c_1} c_1c_1;$ $\quad$ $c_2 : s_1s_1 \succ_{c_2} c_2s_1 \succ_{c_2} s_2s_2 \succ_{c_2} c_2c_2$

We can verify that s_2 satisfies SIC and c_1, c_2 satisfy SIC and WREP, and s_1 arrives in the market in period 2. Also, it can be shown that no matching is dynamically stable with one-sided commitment in this market:

- If $\mu(c_1) = c_1c_1$, then $\mu(c_2) = c_2s_1$ or c_2c_2 or s_2s_2 or s_1s_1. If $\mu(c_2) = c_2s_1$ or c_2c_2, then μ is period-1 blocked by (c_1, s_2) with $\mu'(s_2) = c_1s_2$. If $\mu(c_2) = s_1s_1$, then μ is period-1 blocked by s_1. If $\mu(c_2) = s_2s_2$, then $\mu(s_1) = s_1s_1$ and thus μ is period-1 blocked by (c_2, s_1) with $\mu'(s_1) = s_1c_2$.

- If $\mu(c_1) = s_2c_1$, then $\mu(s_1) = s_1c_2$ to avoid blocking of (s_1, c_2). But again μ is period-2 blocked by (c_1, s_1) with $\overline{\mu}_2(c_1) = s_1$;

- If $\mu(c_1) = s_2s_2$, then μ is period-1 blocked by s_2;

- If $\mu(c_1) = s_1s_1$, then μ is period-1 blocked by s_1;

- If $\mu(c_1) = s_2s_1$, then $\mu(c_2) = c_2c_2$ and thus μ is period-1 blocked by (s_2, c_2) with $\mu'(c_2) = s_2s_2$.

Naturally, the first way is to add symmetric assumptions to S_I. That is, if we require that agents in S_I has preferences that exhibit WREP, then for s_2, $c_1s_2 \succ_{s_2} s_2s_2$ means that c_1c_1 is acceptable, and thus the case $\mu(c_1) = s_2s_2$ may not be blocked any more. Actually, we do have the following existence result.

Theorem 16. (Sufficient Condition 1 for Existence of DSOCBE Matching): In a two-period matching market with arrivals $(S_I, S_{II}, C, \overline{R})$ where everyone $\in S_I \cup C$ has preferences satisfying SIC and WREP, then the set of dynamically stable matchings with one-sided commitment and binding engagements is nonempty.

However, the above assumption will prevent those students from quitting the market in the second period, which may not be that realistic as nowadays, more and more students are planning to begin their own business by dropping out from colleges.

Another idea is to strengthen the asymmetry of assumptions since the blocking power is also asymmetric in our model. The additional assumption for colleges is called **bias toward final outcomes**, whose variant has been used for the property of efficiency. As is suggested by the name, we assume that colleges care more about those who graduate from them, instead those who initially attend them but drop out or transfers to other colleges midway. Other applications include the situation where there is on-job training or an internship in the first period. A more abstract setting is the dynamic principal-agent problem where relationship may change over time and players choose to invest in the first period while the final outcome appears in the second period.

Assumption 5.4 (Bias toward Final Outcome) *For* $c \in C$, $\succ_c$ *presents bias toward final outcome if for* $k, j \in S^c$, $kj \succ_c kc \succ_c cc \Longrightarrow cj \succ_c kc$.
Assumption 5.5. (Weak Sequential Improvement Complementarity, WSIC) $\succ_i$ *satisfies weak sequential improvement complementarity if (i).* $jk \succ_i jj \succ_i ii \Longrightarrow kk \succ_i jj$ *and (ii).* $jk \succ_i ji \succ_i ii \Longrightarrow kk \succ_i ji$.

The second assumption is called weak sequential improvement complementarity since it is actually relaxed from SIC by eliminating those conditions incompatible with arrivals or departures, and thus we do not need to explicitly consider the case of changes in the set of students. Thus, in the following theorem, we show that with certain assumptions on the side of colleges, arrivals and departures of students are consistent with the existence of DSOCBE matchings.

Theorem 17. (Sufficient Condition 2 for Existence of DSOCBE Matching): In a two-period matching market with arrivals $(S, C, \overline{R})$, if $\forall c \in \cup C$ has preferences satisfying SIC, WREP and bias toward the final outcome and $\forall s \in S$, $\succ_s \in WSIC(s)$, then the set of dynamically stable matchings with one-sided commitment and binding engagements is non-empty.

5.4 Relationship to Literature with Overlapping Generations

To end this section, we will provide some comments on the relationship between our paper and the literature concerning one-sided commitment, including those where one-sided commitment is only implicitly included in the exogenous setup of the matching market, such as in Kennes, Monte and Tumennasan (2014a[11], b[12]).

Firstly, in the dynamic setup of this paper, an agent's potential partnerships are defined over the same time frame, which differs from the overlapping generation model used in Kennes, Monte and Tumennasan (2014a[11]), where each child is in demand of daycare services for two periods and the partnership when she is inactive is not defined. For instance, in our finite-horizon model, agents arriving in the last period will only care about their spot partners as the market will terminate afterwards. By comparison, in the overlapping-generations model, agents arriving in the last period will still consider complete partnership plans over the following two periods.[18]

The two modeling techniques fit different real-world problems. Here we provide two examples for our finite-horizon model. Firstly, consider a scenario where the head of some factory faces the decision of assigning two kinds of workers (say, welders and porters/ doctors and nurses) into pairs to complete some task during a day. Workers have pre-determined working schedules showing when they should be on duty. They may either work the whole day or half a day, either in the morning or afternoon. Arrivals and departures are reflected by the exogenous timing of workers and the workers who have cooperated in the morning can only be departed on a mutual consent basis in order to maintain proficiency and privity. The director targets achieving efficiency and dynamic stability. Another example is the school choice problem (college admissions problem) with transfers and dropouts in a certain district (for example, Boston). Students want to complete a six-year study but can attend different schools for different grades. Movements to or from a certain district or even dropouts provide arrivals and departures. For instance, if some student who has just finished her grade 3 studies moves to Boston with her family, then she can be regarded as a entry to the school choice problem in Boston, but she will typically not restart from Grade 1 like an entry in the overlapping generation model. Instead, she will probably continue to attend Grade 4 and that is why the finite-horizon model may be more suitable.

Secondly, perferences are well-defined for both sides of the market in our model, while in the literature, some side (typically the side of institutions like schools) only possesses priorities, which can be determined by the social planner (government) to some extent. Here, we want to examine whether the induced preferences can guarantee good properties in our setup of two-period matching market. For instance, in Kennes, Monte and Tumennasan (2014a[11], the priority system of each shool exhibits the following properties. Firstly, in period 2, the currently enrolled children are given higher priority to all other students, that is, for any $c \in C$, $s_1 \neq s_2 \in S$, $s_1 s_1 \succ_c s_1 s_2$ and $s_1 s_1 \succ_c s_1 c$. Secondly, in period 1, each school behaves according to the spot priority. Equivalently speaking, it just cares about the current student in period 1 instead of full partnership plans. Then for any $c \in C$, $s_1 \neq s_2 \in S$ and $i, j \in S^c \equiv S \cup \{c\}$, $s_1 i \succ_c s_2 j$ if and only if $s_1 s_1 \succ_c s_2 s_2$. Actually, the induced preference of college c can be regarded as lexicographic. When two plans are compared, the period-1 spot partners will be compared firstly. If they are different, then the relative ranking fully depends on the priority of those two individuals. If instead they are identical, then period-2 partners will be further compared. There may be other restrictions, but those

[18] Actually, that also explains why rankability is incompatible with arrivals in period 2 in our model (where the preferences of arrivals are also defined over the first two periods), but compatible with arrivals in the overlapping generations model (where the preferences of arrivals are defined over period 2 and period 3). It is only a conceptual difference.

properties are already enough. For simplicity, we denote that the preferences of colleges **satisfy the assumption of** KKT if they are induced from the priorities defined in Kennes, Monte and Tumennasan (2014a[11]). Then we have the following existence and efficiency result for DSOCBE matchings.

Proposition 3. In a two-period matching market with arrivals $(S, C, \overline{R})$ where every college's preference satisfies the assumption of KKT, and for any $s \in S, c_1, c_3 \in C, c_2 \in C \cup \{s\}$, '$sc_3 \succ_s c_1c_2 \succsim_s c_1c_1 \succ_s ss$ or $sc_3 \succ_s c_1c_2 \succsim_s c_1s \succ_s ss$' $\Longrightarrow c_1c_3 \succ_s c_1c_2$[19], then the set of weakly dynamically stable matchings with one-sided commitment and binding engagements is non-empty.

Intuitively, the assumption of KKT is pretty strong. On the one hand, the first-period matching can be determined based on the spot ranking of students in period 1; on the other hand, since the college prefers a matching in which it matches to the same child for two periods to another matching in which the child is replaced with another in period 2, one-sided commitment will be naturally obeyed as the student can always continue to enroll in the same school which she attended previously. Thus, a mechanism similar to the repeated DA algorithm will guarantee the exsitence of DSOCBE matchings.

Algorithm 7 (RDA) The student-proposing **repeated deferred acceptable algorithm** identifies a matching μ^* as follows:

- *Stage 1. P-DA :*
  Consider a two-period college admission problem $(C, S, \overline{R})$. Define $\overline{R}_1$ as the preferences restricted on certain plans, that is, for agent s, $\overline{R}_1(s)$ agrees with $\overline{R}(s)$ on $X_s^4 \equiv \{cs, cc, sc, ss : c \in C\}$ and all other plans are regarded as unacceptable. Then run the P-DA, which is modified from P-DAFC with X_s^4 replacing X_m^0 as potential proposed plans for the market $(C, S, \overline{R}_1)$ and get the interim matching μ^I .

- *Stage 2. Spot DA:* Define $\overline{R}_2$ as spot preferences conditional on the first-period partnership μ_1^I, that is, $\overline{R}_2(i) = \succ_i^{2(\mu_1^I)}, \forall\ i\ \in S \cup C$. Conduct the spot market S-proposing DA algorithm for $(S, C, \overline{R}_2)$ and get μ_2^{II}.
  Define the ultimate outcome as μ^* where $\forall i \in C \cup S, \mu^*(i) = (\mu_1^I(i), \mu_2^{II}(i))$.

[19]It is easy to show that such an assumption is compatible with arrivals and departures and it can be implied by the assumption of rankability.

6 Discussion

6.1 Matching with No Commitment

We have examined dynamic matching markets with different commitment types– full commitment, two-sided commitment and one-sided commitment, then a natural question is how different the market will be when there is no commitment. Luckily, Kadam and Kotowski (2015a[9], b[10]) have studied the model without commitment thoroughly and in this section, we will briefly overview their setups and results, as well as the comparison with our models.

A dynamic matching market exhibits no commitment if agents matched in the previous periods are completely not obliged to continue the patnership plans. In other words, each agent has the ability to remain single in each period and thus can be regarded as unmatched at the beginning of that period. Moreover, since the set of agents are fixed over time, Kadam and Kotowski (2015a[9]) focuses on the repeated matching problems with a constant pool of individuals.

For a formal definition of dynamic stability without commitment, we directly borrow from Kadam and Kotowski (2015a[9]).

Def 15. (Dynamic Stability with No Commitment, DSNC) *In a two-period matching market* $(M,W,\overline{R})$*, a matching* μ *is* ***dynamically stable with no commitment*** *if*

- μ *is not blocked by any individual. That is,* $\nexists\, i \in M \cup W$ *such that* $ii \succ_i \mu(i)$ *or* $(\mu_1(i), i) \succ_i \mu(i)$.
- μ *is not period-1 blocked by any pair of agents That is,* $\nexists\, m \in M, w \in W$ *such that (i)* $ww \succ_m \mu(m)$ *and* $mm \succ_w \mu(w)$*, or (ii)* $wm \succ_m \mu(m)$ *and* $mw \succ_w \mu(w)$*, or (iii)* $mw \succ_m \mu(m)$ *and* $wm \succ_w \mu(w)$.
- μ *is not period-2 blocked by any pair of agents. That is,* $\nexists\, m \in M, w \in W$ *such that* $(\mu_1(m), w) \succ_m \mu(m)$ *and* $(\mu_1(w), m) \succ_w \mu(w)$.

By comparison, commitment actually means the ability of unilaterally enforcing the first-period partnership to continue in the future and that is where those four models differ from each other. The below summarizes some of the results in Kadam and Kotowski (2015a[9]) under certain assumptions.

Fact 6.1

(1) If agents preferences satisfy (I) SIC, or (II) SIC-(i) and singlehood aversion, which means $ji \succ_i ii \Longrightarrow jj \succ_i ji$, $ij \succ_i ii \Longrightarrow jj \succ_i ij$, then there exists a dynamically stable matching;

(2) If preferences exhibit inertia[20], then every persistent dynamically stable matching is Pareto optimal;
(3) Suppose agents preferences satisfy SIC. If agents can only communicate preferences that satisfy SIC, then the P-DAA is strategyproof for the proposing side.

As is proven in the Appendix A, inertia is actually equivalent to rankability, which implies that the assumptions required for the results in a model without commitment is comparable with (and also similar to) those used in the models with some level of commitment. One the one hand, this suggests that many features of the dynamic matching market should be attributed to the essence of complete partnership plans defined over multiple periods, instead of the specific assumption on the commitment types. One the other hand, it is easy to show that inertia or singlehood aversion is incompatible with arrivals or departures, which explains why all agents are assumed to arrive from the very beginning and stay until the market terminates.

6.2 Welfare Comparisons Across Different Commitment Levels

In the previous sections, four different kinds of commitment are studied as exogeneous parameters in the scheme of two-period matching markets separately. However, as we have discussed, the commitment levels may evolve over time as the social norms change (like the development of marriage markets since fedual times or middle ages), or be modified by the social planners via legislation (like the contract laws in labor markets). Thus, it is worthwhile to compare those different stability notions within the same matching problem in terms of welfare.

For simplicity, we consider the two-period marriage market $(M,W,(T_1,T_2,T_3),\overline{R})$ in Section 4, where T_1, T_2, T_3 respectively denote the set of agents active on the market for two periods, arrive in the market in period 2, and leave the market in period 2. Also, rankability is assumed for individuals in T_1. Our concentration is three kinds of simple comparisons: $\succsim_M$, $\succsim_W$, $\succsim_{M\cup W}$. The first observation is that when termination of relationships is allowed, even in a mutually agreed basis, everyone can be weakly better off. Intuitively, given that individuals may prefer volatile partnership plans, the option of exchanging partners in period 2 without hurting anyone involved will bring about a Pareto improvement.

Proposition 4 In a two-period marriage market $(M,W,(T_1,T_2,T_3),\overline{R})$ where agents in T_1 have rankable preferences, denote μ^{FC} as the outcome of the M-proposing P-DAFC algorithm and μ^{TC}

[20]For concrete definition, one can refer to the appendix A or the original paper directly. Briefly, if $\succ_i$ exhibits inertia, then there is some underlying ranking of inviduals and consistent plans are given extra favoritism.

as the outcome of the M-proposing three-stage TDA algorithm [21], then $\mu^{TC} \succsim_{M\cup W} \mu^{FC}$, that is, $\forall\, i \in M \cup W$, $\mu^{TC}(i) \succsim_i \mu^{FC}(i)$.

Before making other comparisons, it is necessary to clarify that the term 'no commitment' may be a little misleading. By 'no commitment', we do not mean that agents possess no unilateral blocking power except for being unmatched for all periods. In contrast, they simply cannot maintain the previous matching in future periods, and meanwhile, they are entitled with the new right to become unmatched from any specific period. One may regard one-sided commitment as the intermediate between two-sided commitment and no commitment, where both the unilateral blocking powers of remaining single and maintaining previous partership are assigned to the same side of the market. Thus, two-sided, one-sided and no commitment levels are three parallel concepts and none of them is sricter or weaker than another one, which can be reflected in the following comparisons.

Firstly, to compare one-sided and no commitment levels with two-sided commitment respectively, the following example shows that there does not exist an absolute ranking in terms of any of the three relationships $\succsim_M$, $\succsim_W$ or $\succsim_{M\cup W}$. For one-sided commitment, WLOG, suppose that M has the additional blocking power.

Example 6.1 $M = \{m_1, m_2\}, W = \{w_1, w_2\}$ with $T_1 = \{m_2, w_1\}$, $T_2 = \emptyset$, $T_3 = \{m_1, w_2\}$ and the preferences (the preference symbol $\succ$ omitted):

$\mathbf{m_1}: w_1m_1 \quad w_2m_1 \quad m_1m_1;$ $\mathbf{m_2}: w_1w_1 \quad m_2m_2$

$\mathbf{w_1}: m_1m_1 \quad m_1w_1 \quad m_2m_2 \quad w_1w_1;$ $\mathbf{w_2}: m_1w_2 \quad w_2w_2$

Note that the outcome of M-proposing three-stage TDA algorithm is μ^{TC} such that $\mu^{TC}(m_1) = w_2m_1$, $\mu^{TC}(w_2) = m_1w_2$ and $\mu^{TC}(m_2) = w_1w_1$, which is also the unique DSTC matching in the market. Similarly, the results of M-proposing PDA-OC and M-proposing P-DAA coincide at $\mu^{OC} = \mu^{NC}$ such that $\mu^{OC}(m_1) = \mu^{NC}(m_1) = w_1m_1$, $\mu^{OC}(m_2) = \mu^{NC}(m_2) = m_2m_2$, $\mu^{OC}(w_1) = \mu^{NC}(w_1) = m_1w_1$, $\mu^{OC}(w_2) = \mu^{NC}(w_2) = w_2w_2$. Again, μ^{NC} (μ^{OC}) is the unique DSNC (DSOCBE) matching in this market. However, $\mu^{TC} \succ_{m_2} \mu^{OC} = \mu^{NC}$, $\mu^{TC} \succ_{w_2} \mu^{OC} = \mu^{NC}$, while $\mu^{OC} = \mu^{NC} \succ_{m_1} \mu^{TC}$, $\mu^{OC} = \mu^{NC} \succ_{w_1} \mu^{TC}$. Thus μ^{TC} and μ^{NC} (μ^{TC} and μ^{OC}) cannot be ranked according to any of $\succsim_M$, $\succsim_W$ or $\succsim_{M\cup W}$.

Secondly, between one-sided commitment and no commitment, one may reckon that men may be weakly better off in the previous setup since they possess stronger unilateral blocking power. However, with the existence of arrivals, a man in T_2 may become worse off when his desired part-

[21] Here μ^{TC} is dynamically stable with two-sided commitment and binding engagements, DSTCBE. We may need other assumptions to guarantee μ^TC a DSTC matchings.

ner is trapped in a partnership and thus cannot be available in period 2 where there is one-sided commitment. The following example provides some concrete insights.

Example 6.2 $M = \{m_1, m_2\}, W = \{w_1, w_2\}$ with $T_1 = \{m_1, w_1\}$, $T_3 = \emptyset$, $T_2 = \{m_2, w_2\}$ and the preferences (the preference symbol $\succ$ omitted):

$\mathbf{m_1}: w_1w_1 \quad w_1m_1 \quad m_1m_1$; $\qquad \mathbf{m_2}: m_2w_1 \quad m_2w_2 \quad m_2m_2$

$\mathbf{w_1}: m_2m_2 \quad m_1m_2 \quad m_1m_1 \quad w_1w_1$; $\qquad \mathbf{w_2}: w_2m_2 \quad w_2w_2$

The unique DSOCBE matching is μ^{OC} such that $\mu^{OC}(m_1) = w_1w_1$, $\mu^{OC}(m_2) = m_2w_2$, $\mu^{OC}(w_2) = w_2m_2$ while the unique DSNC matching is μ^{NC} such that $\mu^{NC}(m_1) = w_1m_1$, $\mu^{NC}(m_2) = m_2w_1$, $\mu^{NC}(w_1) = m_1m_2$, $\mu^{NC}(w_2) = w_2w_2$. Moreover, $\mu^{NC} \succ_{m_2} \mu^{OC}$, $\mu^{NC} \succ_{w_1} \mu^{OC}$, $\mu^{OC} \succ_{m_1} \mu^{NC}$, $\mu^{OC} \succ_{w_1} \mu^{NC}$. Thus μ^{OC} and μ^{NC} cannot be ranked according to any of $\succsim_M$, $\succsim_W$ or $\succsim_{M\cup W}$.

6.3 Spot Mechanism: Full Commitment as an Example

Recall that in most of the algorithms used above, it is required that everyone's full preference defined over complete partnership plans is reported before the market actually starts, so that the entire matching is determined prior to period 1. For example, in PDA-FC used in the model with full commitment, agents should make engagements (sign contracts), which are proposed long before the date of marriage (when the terms are put into effect). This may induce some concerns about the feasibility of the prospective proposals, especially when one has to propose to someone w who does not exist on the market at present. Thus, it is necessary to figure out whether it is possible to come up with a stable mechanism that only depends on spot preferences. If so, the central planner may simply make use of the repeated DA algorithm with the spot rankings defined over individuals instead of complete partnership plans. However, thanks to the properties unique to a multi-period model, the beautiful and simple candidate for the stable mechanism fails. We will briefly clarify the idea using the model with full commitment as an example.

For a two-period matching market (M_1, M_2, W_1, W_2, R) where $M = M_1 \cup M_2$, $W = W_1 \cup W_2$, a spot rule, or a mechanism based on spot markets means that only spot preferences in period 1 can influence μ_1 and only spot preferences in period 2 can influence μ_2 given μ_1 fixed and known. Equivalently, regardless of all the arrivals and departures in period 2, μ_1 should not be influenced. Formally we have the following definition of spot rules.

Def 16. (Spot Rule) *Denote the set of two-period marriage markets as* Θ. $\phi = (\phi_1, \phi_2)$ *is a* ***spot rule*** *defined over* Θ, $\phi(\theta) = \mu : (M \cup W) \to (M \cup W)^2$ *is a two-period matching on the market* θ

for any $\theta = (M_1, M_2, W_1, W_2, R) \in \Theta$, where μ is determined in the following ways:
(i). $\mu_1|_{M_1 \cup W_1}$ only depends on the submarket $(M_1, W_1, R|_{M_1 \cup W_1})$ and $\forall\ i \in (M - M_1) \cup (W - W_1)$, $\mu_1(i) = i$.
(ii). Denote $\Gamma_1 = \{i \in M_1 \cup W_1 : \mu_1(i) \neq i\}$, then $\mu_2|_{M_2 \cup W_2 - \Gamma_1}$ only depends on the submarket $(M_2 - \Gamma_1, W_2 - \Gamma_1, (\succsim_i^{2(\mu_1)})_{i \in M_2 \cup W_2 - \Gamma_1})$ where $\succsim_i^{2(\mu_1)}$ is the spot preference in period 2 given μ_1 for agent i, and $\forall\ i \in (M - M_2) \cup (W - W_2)$, $\mu_2(i) = i$.
(iii). For $i \in \Gamma_1$, $\mu(i) = (\mu_1(i), \mu_1(i))$, and for $i \in (M \cup W - \Gamma_1)$, $\mu(i) = (i, \mu_2(i))$.

If some spot rule can produce a DSFC matching in any two-period marriage market $\theta \in \Theta$, then the spot rule is defined to be dynamically stable with full commitment. If such a spot rule exists, then agents only need to report their spot preferences for the corresponding periods and the central planner is benefited by utilizing the same simple spot mechniasm in both the dynamic and static markets. Unfortunately, the conjecture is not true.

Proposition 5 There does not exist a DSFC spot rule in the general preference domain.

Intuitively, in any feasible spot rule that is stable in the one-period market, the incentives of period-1 agents to stay unmatched currently and wait for a better partner arriving in period 2 are ignored since those arrivals will not be taken into account in determining the period-1 matching. Thus, one possible method to guarantee the the existence of DSFC spot rule is to assume away the motivation of strategic waiting. For example, recall the assumption of **strong impatience (Def. 11)** used in the model with two-sided commitment, which states that for an agent *m* in M_1 (*w* in W_1), if a consistent plan *ww* (*mm*) is acceptable, then *m* (*w*) will prefer it to any plan where he (she) is single in period 1 and matched to $w_1 \in W_2 - W_1$ ($m_1 \in M_2 - M_1$). That is, the cost of being unmatched at present outweighs the gains from waiting for a good choice in the future, which is reasonable given a relatively large discount factor or the existence of present bias.

Additionally, the restriction of strong impatience is not enough since it says nothing about the unravelling plans between agents in $M_1 \cup W_1$. For example, the spot rule may pair two agents with a consistent plan, while they can prefer to postpone the beginning of partnership plan to the next period, which again ruins the stability of the spot rule. In this sense, we add an extra condition called weak impatience, although it is not implied by the strong impatience.

Assumption 6.1 (Weak Impatience, WI) For $m \in M_1 \cap M_2$, $\succ_m$ satisfies weak impatience if for any $w \in W_1 \cap W_2$, $ww \succ_m mw$. For $w \in W_1 \cap W_2$, $\succ_w$ satisfies weak impatience if for any $m \in M_1 \cap M_2$, $mm \succ_w wm$.

There are some observations worth mentioning about weak impatience. Firstly, rankable preferences (and preferences with SIC) naturally exhibit weak impatience and that is why it is not explicitly required in previous models. Secondly, if we care about weak stability where agents arriving in period 2 cannot form a blocking pair with those matched in period one, then weak impatience can guarantee the existence of weakly dynamic stability with full commitment (WDSFC). Furthermore, with strong impatience, the WDSFC matching can be upgraded to a DSFC matching. That is why we name those two assumptions in the above way.

Proposition 6 In a two-period marriage market (M_1, M_2, W_1, W_2, R) where agents in $(M_1 \cap M_2) \cup (W_1 \cap W_2)$ have preferences that exhibit weak impatience and strong impatience, then set of DSFC matchings in nonempty. Moreover, any DSFC matching is efficient among matchings with full commitment and the repeated DA algorithm with full commitment (to be defined as below) is strategy-proof for the proposing side.

The existence of DSFC matching is proved using a modification of the RDA algorthm by forcing those agents matched in period 1 to leave the market immediately.

Algorithm 8 (RDA-FC) The student-proposing **repeated deferred acceptable algorithm with full commitment** identifies a matching μ^* as follows:

- *Stage 1. Spot DA in period 1:*
  Carry out the M-proposing DA algorithm for the period-1 spot market $(M_1, W_1, \{\succsim_i^1\})$ where $k \succsim_i^1 j$ if and only if $kk \succsim_i jj$. Denote the result matching as μ_1 and $E_1 \equiv \{i \in M_1 \cup W_1 : \mu_1(i) \neq i\}$.
- *Stage 2. Spot DA for those unmatched in period 1:*
  Carry out the M-proposing DA algorithm for those unmatched in μ_1 in the period-2 spot market $(M_2 \cup M_2 - E_1, W_1 \cup W_2 - E_1, \{\succsim_i^2\})$ where $k \succsim_i^2 j$ if and only if $ik \succsim_i ij$. Deote the result matching as μ_2
  Then the ultimate result μ^* is defined by $\forall i \in E_1, \mu^*(i) = (\mu_1(i), \mu_1(i))$ and $\forall i \in M \cup W - E_1$, $\mu^*(i) = (i, \mu_2(i))$.

6.4 Extension to Multiple Periods

In the main part of this paper, we focus on the two-period matching market since it can reflect most of unique characteristics of dynamic problems which are not shared by static ones. Here we will briefly show how to extend the model into multiple (still finite) periods using the examples of no commitment and two-sided commitment.

Consider a T-period matching market $((M_1,M_2,\dots M_T),(W_1,W_2,\dots W_T),R)$ for some integer number T, $M_t \cup W_t$ denotes the set of agents available for matching in period $t = 1,2\dots T$. Denote $M \equiv \bigcup_{t=1}^{T} M_t$, $W \equiv \bigcup_{t=1}^{T} W_t$. In R, for any $i \in M \cup W$, $\succsim_i^0$ is defined on partnership plans over the periods when i is active, that is, $\{t : i \in M_t \cup W_t\}$. For simplicity of notations, we can consider the extended T-period matching market $(M,W,\overline{R})$ where $\succsim_i$ is defined on complete partnership plans in $(M\cup W)^T$ and i will regard all plans where he or she is matched in some $t' \in \{t : i \notin M_t \cup W_t\}$ as unacceptable.

A partnership plan for $m \in M$ is denoted as $x = (x_1,\cdots,x_T) \in (W^m)^T$ where $W^m \equiv W \cup \{m\}$. With no confusion, x is simplified as $x = x_1\cdots x_T$. Similarly we can define the partnership for $w \in W$. Further, denote $x_{\geqslant t} = x_t \cdots x_T \in (M\cup W)^{T-t+1}$, $x_{>t} = x_{t+1}\cdots x_T \in (M\cup W)^{T-t}$, $x_{\leqslant t} = x_1 \cdots x_t \in (M\cup W)^t$, $x_{<t} = x_1 \cdots x_{t-1} \in (M\cup W)^{t-1}$. Then $x = (x_{<t}, i, j, x_{>t})$ if i and j are respectively the partners for period t and $t+1$. Moreover, x^{jk} denotes the partnership plan where $x_t \in \{j,k\}$ for any $t = 1,\dots,T$, which will be used to represent the plan that can be enforced by a pair of individuals. As a special case $x^i = i\cdots i \in (M\cup W)^T$. The definition of spot matching in period t coincides with that in a static market. Then $\mu = (\mu_1,\cdots,\mu^T) : (M\cup W) \to (M\cup W)^T$ is a T-period matching if μ_t is a spot matching in period t for any $t = 1,\dots,T$. $\mu_{\geqslant t}, \mu_{>t}, \mu_{\leqslant t}$ and $\mu_{<t}$ are then defined correspondingly.

For a model with full commitment, which implies that breakups of existing relationships cannot happen, we define that a partnership plan x for agent i **exhibits full commitment** iff $x_t \neq i$ for some $t = 1,\dots,T \Longrightarrow x_{t'} = x_t$ for any $t' \geqslant t$.

Def 17. (T-period Dynamic Stability with Full Commitment, DSFC) *A matching μ on the T-period matching market $(M,W,\overline{R})$ is dynamically stable with full commitment if*

1. *For any $i \in M\cup W$, the partnership plan $\mu(i)$ exhibits full commitment;*
2. *μ is not blocked by any individual, that is, $\nexists i \in M\cup W$ such that $x^i = i\cdots i \succ_x \mu(x)$;*
3. *μ is not blocked by any pair of agents with full commitment in period t, that is, $\nexists\, m \in M$ and $w \in W$ and some matching μ' defined over $\{m,w\}$ that exhibits full commitment, such that $\mu'(m) \succ_m \mu(m)$ and $\mu'(w) \succ_w \mu(w)$.*

Intuitively, under the restriction of full commitment, agents only care about the first period when they are matched, which obscures the difference between two periods and multiple periods. Thus, we have the following result for the existence of DSFC matchings using a natural extension of the two-period PDA-FC alogrithm.

Proposition 7 The set of dynamically stable matchings with full commitment is nonempty for any T-period matching market.

Algorithm 9 (T-period PDA-FC) The T-period man-proposing plan deferred acceptable algorithm identifies a matching μ^* as follows:
1). For each $m \in M$, let $X_m^0 \equiv \{x \in (W^m)^T : x \text{ exhibits full commitment }\}$. Initially, no plans in X_m^0 have been rejected;
2). In round $\tau \geqslant 1$,
(a). Let $X_m^\tau \subset X_m^{\tau-1}$ be the set of plans that have not been rejected in the previous rounds and m propose the most preferred plan in X_m^τ. Proposing mm implies that agent m has been rejected by any acceptable plans involving a women.
(b). Let X_w^τ denote the set of plans proposed to w in round τ. If $x^w \succ_w x$ for all $x \in X_w^\tau$, then w rejects all the proposals. Otherwise, w keeps her most preferred plan in X_w^τ tentatively and rejects all others.
3). Repeat procedure 2) until no rejections occur. If w ends up keeping m's proposal in the final round, define $\mu^*(m)$ and $\mu^*(w)$ accordingly. If i does not make or keep any proposal in the final round, let $\mu^*(i) = ii$.

Now we turn to the model with two-sided commitment. Denote $\mu_0(i) = i$ for all $i \in M \cup W$ as the ex-ante spot matching, which implies that everyone is unmatched before the market begins. On the one hand, in any period, each agent have the right to unilaterally enforce the partnership in the last period to persist. In other words, agent i **period-t blocks** a matching μ if $(\mu_{<t}(i), x_{\geqslant t}^{\mu_{t-1}(i)}) \succ_i \mu(i)$. Specifically, i period-1 blocks μ if $x^i \succ_i \mu(i)$. Then a matching is **individually rational** if it is not blocked by any individual.

On the other hand, for pairwise or coalitional blocking, we follow the trandition of the two-period model. In the first period, **a pair** (m,w) **can period-1 block** μ if there exists a μ' defined over $\{m,w\}$, which is itself individually rational, such that $\mu'(m) \succ_m \mu(m)$ and $\mu'(w) \succ_w \mu(w)$. [22] Like the definition of two-period market, when w (or m) is considering whether to form a blocking pair with m (or w), she rationally expects that any μ' blocked by m (or w) will not actually happen,

[22] It is true that in the definition of pairwise period-1 blocking, the blocking matching may be not mutually acceptable, but if this happens, the orignal matching itself is not individually rational. Actually, as our focus is the set of DSTC matchings, the two-period setup can be regarded as a sepcial case of the general T-period setup.

since m (or w) can always unilaterally enforce a consistent plan beginning at any period. Thus with the implicit assumption that the feature of two-sided commitment is common knowledge, it is required that the blocking matching is itself not blocked by any individual.

In any period $t \geqslant 2$, coalitional matching is more appropriate since agents may be matched in μ_{t-1} and two-sided commitment provides them with the right to maintaining the partnership. Thus when some m with $\mu_{t-1}(m) = w$ wants to change the parnter in period t, then such blocking is only feasible with the consent of w. That is, w should also be involved in the blocking set and matched to a more preferred plan. Accordingly, a **coalition $S \subset M \cup W$ can period t block μ** if

i Mutual involvement: $\forall\, i \in S$, if $\mu_{t-1}(i) \neq i$, then $\mu_{t-1}(i) \in S$;

ii Implementation and mutual benefits: $\exists\, \overline{\mu}_{\geqslant t} : S \to S^{T-t+1}$, such that $(\mu_{<t}, \overline{\mu}_{\geqslant t})$ is individually rational, $\overline{\mu}_{\geqslant t}(i) = x^{ij}_{\geqslant t}$ for some $j \in A$, and $(\mu_{<t}, \overline{\mu}_{\geqslant t}) \succ_S \mu$.

Notice that in period-t, when forming the blocking coalition, each agent cannot be matched to two or more different individuals on the opposite side of the market in the blocking matching for the rest $T-t+1$ periods. This reveals that our notion of stability (to be defined later) is an intermediate concept between pairwise stability and core. Actually, we allow for the blocking set involving more than two agents just because of the requirement of mutual involvement, instead of more volatile blocking matchings. In this sense, the definition resembles pariwise blocking more.

Now we can have the following general definition of dynamic stability with two-sided commitment.

Def 18. (T-period Stability with Two-sided Commitment, DSTC) *In a T-period marriage market $(M, W, \overline{R})$, a matching μ is **dynamically stable with two-sided commitment** if it is individually rational, not period-1 blocked by any pair of agents and not period-t blocked by any coalition for any $t = 2, \ldots, T$.*

7 Conclusion

This paper presents three different but related settings to provide intuition in modelling the multi-period one-to-one matching market with different levels of commitment, which reflects the restrictive power of existing relationships and the cost of breakups, as a contrast to the setup with no commitment in Kadam and Kotowski (2015a)[9].

The baseline model with full commitment is applicable when the matching is somehow once-and-for-all and agents will leave the market if and only if they get matched at some period. Typically, no restrictions other than strictness about preferences are required to obtain the existence of dynamically stable matching with full commitment and other results about Pareto-efficiency, one-sided

strategy-proofness and lattice structure, while the difference from the static market lies in the difficulty of the existence of a dynamically stable spot rule like the repeated DA algorithm.
By comparison, our setting with two-sided commitment, or peaceful divorce, allows for agreed breakup of present relationships and applies in the cases where contracts are legally binding or the cost of forceful divorce (ex. liquidated damages) dominates the possible gains from divorce. Although the set of stable matchings with two-sided commitment is nonempty with a fixed set of market participants over time, additional conditions are necessary to introduce arrivals and departures while maintaining the existence of dynamic stability. One approach is to relax the requirements of stability, such as limiting the blocking power of agents only appearing in period 2, or allowing for binding unravelling contracts. The other approach is to add more assumptions on preferences like strong impatience.
Finally, for markets with one-sided commitment such as the labor market and school choice problem, where the structure is typically designed in favour of workers and students by assigning them more unilateral blocking power, results similar to those obtained in the two-sided commitment case hold.

For welfare comparisons, we show that under reasonable conditions, the market with two-sided commitment can weakly Pareto dominate the market with full commitment, as mutually beneficial rematches can be carried out in the former case. However, there is no absolute rankings in terms of the preferences of the set of men $\succsim_M$ or of the set of women $\succsim_W$ between other types of commitment.

Future work may include three possible directions. Firstly, we can have more detailed discussion about the conditions to guarantee the existence of dynamic stability and other desirable properties and polish them according to the characteristics of real-world applications. Secondly, extensions to incorporate monetary transfers and uncertainty can be conducted to assess whether the level of commitment has a different impact on the market. Typically, a cardinal model is more appropriate to deal with those issues. Finally, it may be worthwhile to consider stable mechanisms which are not based on the deferred acceptance algorithm for the models with different levels of commitment, and then compare them with those adopted in this paper. Potentially better mechanisms are around the corner.

A Appendix: Notions and Assumptions of Dynamic Stability

For the multi-period matching market of our concern, there are majorly the four kinds of notions of stability and corresponding assumptions in the literature as below:

(1). Time-separability and time-invariance: core, recursive core, self-sustaining stability, strict self-sustaining stability ... (Damiano and Lam, 2005[5]);
(2). Substitutability and history-independence: pairwise stability. (Bando, 2012[3]);
(3). Strong rankability, rankability: autarkic stability, stability (Kennes, Monte and Tumennasan, 2014a[11], b[12]);
(4). Reflects a spot ranking, Inertia, SIC: ex-ante stability, dynamic stability (Kadam and Kotowski, 2015a[9], b[10]).

A.1 Definitions of Assumptions

We consider the one-to-one two-sided two-period matching market (M,W,P). With symmetry, we only need to consider the preference for the generic man $m \in M$.

- **Time-separability:** $\succ_m$ defined on $(W \cup \{m\})^2$ is time-separable if $\exists$ utility indices v_{1m} and v_{2m} defined over $W \cup \{m\}$ such that $\forall x_1, x_2, y_1, y_2 \in W \cup \{m\}$,

$$(x_1,x_2) \succ (y_1,y_2) \Longleftrightarrow v_{1m}(x_1) + v_{2m}(x_2) > v_{1m}(y_1) + v_{2m}(y_2);$$

- **Time-invariance:** Besides time-separability,$\succ_m$ is time-invariant if $\forall x \in W \cup \{m\}$, $v_{1m}(x) = v_{2m}(x) \equiv v_m(x)$;

- **Responsiveness:** $\succ_m$ is responsive if $\forall x_1, x_2 \in W \cup \{m\}, x_1x_1 \succ_m x_2x_2 \Longleftrightarrow yx_1 \succ_m yx_2$ and $x_1y \succ_m x_2y$, $\forall y \in W \cup \{m\}$;

- **Strong Rankability:** $\succ_m$ satisfies strong rankability if $\forall x_1, x_2 \in W \cup \{m\}, x_1x_1 \succ_m x_2x_2 \Longleftrightarrow yx_1 \succ_m yx_2$ and $x_1y \succ_m x_2y$, $\forall y \in W \cup \{m\}$;

- **Rankability:** $\succ_m$ satisfies rankability if $\forall x_1, x_2 \in W \cup \{m\}, x_1x_1 \succ_m x_2x_2 \Longleftrightarrow yx_1 \succ_m yx_2$ and $x_1y \succ_m x_2y$, $\forall y \in W \cup \{m\}$ and $y \neq x_2$.[23]

- **Substitutability:** Consider $Ch_m(A)$ as the **choice function** of agent m for $A \subseteq (W \cup \{m\})^2$ such that $Ch_m(A) \in A$ and $Ch_m(A) \succeq xy$, $\forall xy \in A$. Then m has substitutable preferences if $w \in W$ and $A \subseteq (W \cup \{m\})^2$, we have
(i). $w = Ch_m^{t=1}(A) \Longrightarrow Ch_m^{t=2}(A) = Ch_m^{t=2}(A')$ for $A' = A \cap ((W \cup \{m\} \backslash \{w\}) \times (W \cup \{m\}))$;
(ii). $w = Ch_m^{t=2}(A) \Longrightarrow Ch_m^{t=1}(A) = Ch_m^{t=1}(A')$ for $A' = A \cap ((W \cup \{m\}) \times (W \cup \{m\} \backslash \{w\}))$;

- **Reflecting a spot ranking:** $\succ_m$ reflects the spot ranking P_m if $j_tR_mk_t$, for all t and $j_tP_mk_t$ for some t =1 or 2, then $j_1j_2 \succ_m k_1k_2$.

[23] In the definition of rankability and strong rankability, $\Longleftarrow$ can be directly proved by way of contradiction. Thus, to show those assumptions hold, it suffices to show that the $\Longrightarrow$ part.

- **Inertia:** $\succ_m$ exhibits inertia relative to some preference $\succ'_m$ if (i). $jj \succ'_m kl \Longrightarrow jj \succ_m kl$; (ii). $jj \succ'_m kk \Longleftrightarrow jj \succ_m kk$; and (iii). If $j \neq j'$, $k \neq k'$, then $jj' \succ'_m kk' \Longleftrightarrow jj' \succ_m kk'$. *Later, when we say that $\succ_m$ exhibits **inertia**, we implicitly mean that it has inertia relative to some matching that reflects a spot ranking.*

- **Sequential improvement complementarity (SIC)**: $\succ_m$ satisfies SIC if
(i). $jk \succ_m jj \succeq_m mm \Longrightarrow kk \succ_m jj$;
(ii). $jk \succ_m jm \succ_m mm \Longrightarrow kk \succ_m jm$;
(iii). $mk \succ_m mj \succeq_m mm \Longrightarrow kk \succ_m mj$.

A.2 Relationship among Assumptions and Notions of Stability

Lemma 1 : Consider a one-to-one two-sided two-period matching market (M,W,P). Assume that every agent has strict preferences, then
(1). Time-separability + Time-invariance $\Longrightarrow$ $\succ_i$ reflects a spot ranking $\Longrightarrow$ inertia;
(2). $\succ_i$ reflects a spot ranking $\Longleftrightarrow$ Responsiveness $\Longleftrightarrow$ Strong Rankability $\Longrightarrow$ Rankability;
(3). Rankability $\Longleftrightarrow$ Inertia;

Proof. With symmetry, we only need to prove the lemma for the generic man $m \in M$.We will prove each chain of relationships from left to right.
(1). **Step 1**: If $\succ_m$ is time-separable and time-invariant, that is, $\exists$ a utility index v_m defined over $W \cup \{m\}$ such that the utility function over a specific matching μ is $u_m(\mu) = v_m(\mu_1(m)) + v_m(\mu_2(m))$. Define that the spot ranking P_m is reflected by v_m, that is $jP_mk \Longleftrightarrow v_m(j) > v_m(k)$. If $j_tR_mk_t$, for all t and $j_tP_mk_t$ for some t =1 or 2, then we know $v_m(j_t) \geqslant v_m(k_t)$, for all t and $v_m(j_t) > v_m(k_t)$ for some t =1 or 2. Accordingly, $v_m(j_1) + v_m(j_2) > v_m(k_1) + v_m(k_2)$ and $j_1j_2 \succ_m k_1k_2$ and thus $\succ_m$ reflects the spot ranking defined by the utility index.
However, the opposite is incorrect since time-separability and time-invariance will specify the comparison between plans j_1j_2 and k_1k_2 where $j_1P_mk_1$ and $k_2P_mj_2$ for any given utility index, while reflecting a spot ranking is ambiguous in such cases.
Step 2: Recall the definition of inertia and it is straightforward that a preference relation that reflects some spot ranking exhibits inertia relative to itself.
(2). **Step 1: $\succ_m$ reflects a spot ranking $\Longleftrightarrow$ Responsiveness**
For sufficiency, assume that $\succ_m$ reflects a spot ranking denoted as P_m. By definition, $\forall x,y \in W \cup \{m\}$, $xx \succ_m yy \Longleftrightarrow xP_my$. Then, notice that $\forall z \in W \cup \{m\}$, zR_mz and thus by definition of reflecting a spot ranking P_m, $xP_my \Longleftrightarrow xz \succ_m yz$ and $zx \succ_m zy$. Thus $\succ_m$ is responsive.
For necessity, define the spot ranking P_m reduced by $\succ_m$ as $xP_my \Longleftrightarrow xx \succ_m yy$. WLOG, assume that $x_1P_my_1$ and $x_2R_my_2$, then equivalently $x_1x_1 \succ_m x_2x_2$ and $y_1y_1 \succeq y_2y_2$. Responsiveness implies

that $x_1x_2 \succ_m y_1x_2 \succeq y_1y_2$ and thus $\succ_m$ reflects P_m.

Step 2: Responsiveness $\Longleftrightarrow$ Strong Rankability

Definitions completely the same, while rankability has been used in one-to-one, multi-period matching markets more and responsiveness more in many-to-one, single-period matching markets.

Step 3: Strong Rankability $\Longrightarrow$ Rankability

Directly by definition. The former one has more restrictions and thus narrow the set the qualified preferences.

(3). For necessity, assume that $\succ_m$ exhibits inertia relative to $\succ'_m$, which reflects a spot ranking P_m. By definition, $kk \succ_m jj \Longleftrightarrow kk \succ'_m jj \Longleftrightarrow kP_mj$. For $l \in W \cup \{m\}$ and $l \neq j$, jl is an inconsistent plan, while kl may either be consistent or not. Recall that the only change induced by inertia is that consistent plans are further favoured over inconsistent plans, then we have $kl \succ_m jl$. Similarly, when $l \neq j$, we can show $lk \succ'_m lj \Longrightarrow lk \succ_m lj$. Now necessity holds.

For sufficiency, it suffices to find a $\succ'_m$ which reflects a spot ranking such that $\succ_m$ exhibits inertia relative to $\succ'_m$.

Firstly, once again define P_m as the spot ranking induced by $\succ_m$;

Then, find the corresponding $\succ'_m$. Note that for a given spot ranking, there exists a class of preferences that reflect it and we are now talking about existence. Denote that j_1j_2 and k_1k_2 are *comparable w.r.t the spot ranking* if either $j_tR_mk_t$ for all t, or $k_tR_mj_t$ for all t. For those pairs of plans comparable w.r.t P_m, define $\succ'_m$ as suggested by the definition of rankability; for the other cases, define $\succ'_m = \succ_m$.

Now we show that $\succ_m$ exhibits inertia relative to $\succ'_m$ by considering the three conditions separately for the two groups of comparisons mentions above.

A). For comparable j_1j_2 and k_1k_2 w.r.t P_m.

(i). If $jj \succ'_m kl$ and $k \neq l$, the assumption of comparison indicates that jR_mk and jR_ml, and at least one of them is strict. (**WLOG, we always assume that the first period spot ranking is strict when such a situation occurs in the following proof for simplicity.**) Then $jj \succ_m kk$ and $jj \succeq_m ll$. From rankability, $jj \succ_m kj \succeq_m kl$ and transitivity implies $jj \succ_m kl$;

(ii). $jj \succ'_m kk \Longleftrightarrow jP_mk \Longleftrightarrow jj \succ_m kk$;

(iii). "$\Longrightarrow$": If $jj' \succ'_m kk'$ and $j \neq j'$, $k \neq k'$. Two possible cases with comparability.

If kP_mj and $k'R_mj'$, $kk' \succ'_m jj'$ since $\succ'_m$ reflects the spot ranking P_i. Contradiction!

If jP_mk and $j'R_mk'$, then by definition of P_m, we have $jj \succ_m kk$ and $j'j' \succ_m k'k'$. Rankability and $k \neq k'$ further imply that $jj' \succeq_m jk' \succ_m kk'$, that is, $jj' \succ_m kk'$.

"$\Longleftarrow$": If $jj' \succ_m kk'$ and $j \neq j'$, $k \neq k'$. Two possible cases with comparability.

If kP_mj and $k'R_mj'$, then by definition of P_m, we have $kk \succ_m jj$ and $k'k' \succ_m j'j'$, and $kk' \succ_m jj'$ since $\succ_m$ satisfies rankability and $j \neq j'$. Contradiction!

If jP_mk and $j'R_mk'$, then $jj' \succ'_m kk'$ since $\succ'_m$ reflects P_m.
B). If j_1j_2 and k_1k_2 are not comparable w.r.t spot ranking, then by definition we have $\succ_m = \succ''_m$ over those comparisons. Thus the three conditions are naturally satisfied. This completes the proof. □

B Appendix: Proofs

Proof of Theorem 1: It suffices to prove that μ^* is dynamically stable with full commitment. Firstly, the algorithm will terminate since the number of proposals are finite and new proposals have to be rejected to let the algorithm persist. Secondly, the outcome is a matching with full commitment since only proposals with consistent plan or engagement can be made and accepted, and μ^* is only determined by the proposals from one side. Thirdly, μ^* is individually rational since men will only proposal acceptable plans and women will only keep acceptable plans. Fourthly, μ^* is not blocked by any pair of agents. Suppose by way of contradiction that (m,w) blocks μ^*, then either (i). $ww \succ_m \mu(m)$ and $mm \succ_w \mu(w)$ or (ii). $mw \succ_m \mu(m)$ and $wm \succ_w \mu(w)$. If (i) holds, then m must have proposed to w with the plan ww but got rejected in some round, either because ww is unacceptable to w or w has received a better proposal. Notice that women will be weakly better off as the algorithm proceeds and thus $\mu^*(w) \succ_w mm$, which is a contradiction. Similar argument can prove that (ii) is also impossible. This completes the proof. □

Proof of Theorem 2: It suffices to show that the outcome matching μ^M of the M-proposing P-DAFC is M-optimal, or equivalently speaking, no man is ever rejected by any achievable plan in any round τ. We then prove the result by induction on τ.
(1). For $\tau = 1$, every man proposes to his top choice in X_m^0. If m makes the proposal mm, then no other partnership is acceptable to m and m will remain single in two periods in all stable matchings. Thus m will never be rejected by any achievable plans since the only acceptable plan is mm. If m does not propose to himself, then $\exists\ (y_0,w_0) \in X_m^0$ such that $y_0w_0 \succeq_m yw$ for all $w \in W$ and y=w or m and m proposes to w_0 with y_0w_0 but gets rejected. This implies that $z_0m \succ_{w_0} w_0w_0$, where z_0 is the **reflection** of y_0 in that $y_0 = w_0 \Longleftrightarrow z_0 = m$ and $y_0 = m \Longleftrightarrow z_0 = w_0$. Suppose by way of contradiction that y_0w_0 is achievable for m, that is, $\exists$ a dynamically stable matching with full commitment μ' s.t. $\mu'(m) = y_0w_0$ and thus z_0m is acceptable for w_0 where z_0 reflects y_0. Consider in the algorithm w_0 rejects z_0m in round 1, then $\exists\ m_1 \in M$ s.t. $\exists\ y_1 = m_1$ or w_0 and y_1w_0 is agent m_1's top choice in $X_{m_1}^0$ and $z_1m_1 \succ_w z_0m$ where z_1 is the **reflection** of y_1. $\mu'(m) = y_0w_0$ implies $\mu'(m_1) \neq y_1w_0$ and from strictness of $\succ_{m_1}$, we have $y_1w_0 \succ_{m_1} \mu'(m_1)$. Recall that $z_1m_1 \succ_w z_0m = \mu'(w_0)$, and thus (m_1,w_0) blocks μ' with m_1 being matched to y_1w_0. Then μ' is not dynamically stable, which is a contradiction. This shows that y_0w_1 is not achievable for agent m.
(2). Assume that in all rounds from 1 to $\tau - 1$, no man is rejected by an achievable plan. Suppose

that in round τ, m proposed to w with y_0w where $y_0 = m$ or w and w rejects the corresponding z_0m where z_0 is the **reflection** of y_0. Suppose by way of contradiction that y_0m is achievable for m, that is, $\exists$ a dynamically stable matching μ'' s.t $\mu''(m) = y_0w$. Then z_0m must be acceptable to w and w rejects z_0m in favour of another plan from another agent $m_1 \in M$ and m_1 proposed to w with y_1w in round τ or previously. Based on the assumption, m_1 is not rejected by any achievable plan and thus for any achievable plans x_1x_2 for agent m_1 with $x_1x_2 \neq y_1w$, we have $y_1m \succ_{m_1} x_1x_2$. Again, $\mu''(m) = y_0w$ implies $\mu''(m_1) \neq y_1w$ and from strictness of $\succ_{m_1}$, we have $y_1w \succ_{m_1} \mu''(m_1)$. Meanwhile, w rejects z_0m in favour of z_1m_1 suggests $z_1m_1 \succ_w z_0m = \mu''(w)$, where z_1 reflects y_1. Thus (m_1, w) blocks μ with m_1 being matched to y_1w, a contradiction!
Accordingly, in any round τ, all men are not rejected by any achievable plans and the theorem follows. $\square$

Proof of Theorem 3: Due to symmetry, we only need to show that if μ, μ' are dynamically stable with full commitment, then $\mu \succ_M \mu' \Longrightarrow \mu' \succ_W \mu$. Suppose by way of contradiction that this is not true, then there must be some woman $\overline{w}$ s.t. $\mu(\overline{w}) \succ_{\overline{w}} \mu'(\overline{w}) \succeq_{\overline{w}} \overline{w}\overline{w}$. Thus, $\exists m \in M$, $z = m$ or $\overline{w}$, s.t $\mu(\overline{w}) = zm$ and $\mu(m) = y\overline{w}$ where y is the reflection of z. Notice that $\mu'(\overline{w}) \neq \mu(\overline{w}) = zm$ $\Longrightarrow \mu'(m) \neq \mu(m) = y\overline{w}$. Since $\mu \succ_M \mu'$ and preferences are strict, $\mu(m) = y\overline{w} \succ_m \mu'(m)$. Recall that $zm = \mu(\overline{w}) \succ_{\overline{w}} \mu'(\overline{w})$ and thus $(m, \overline{w})$ blocks μ' with m being matched to $y\overline{w}$, a contradiction!
$\square$

Proof of Theorem 4: (1). The impossibility result can directly come from the Theorem 4.4 in Roth and Sotomayor (1990). Specifically, we can assume that all agents come to the market in the second period and the dynamic matching market will degenerate to a static one. Then the counterexample in Roth and Sotomayor (1990) is still valid here.
(2). We need a lemma for our proof.
Lemma C.1 Suppose μ is individually rational with full commitment, let M' be the set of men who prefer μ to μ^M. If M' is nonempty, then there is a pair (m, w) that block μ such that $m \in M - M'$ and $w \in \mu_2(M')$.
Proof of Lemma C.1
Case 1: If $\mu_2^M(M') = \mu_2(M') \equiv W'$. Denote w_0 as the last one in W' to receive an acceptable plan from someone in M' in M-proposing P-DAFC. Then w_0 will not reject this offer, otherwise the corresponding $m \in M'$ will again make an acceptable offer to some $w \in W'$ as $\phi_\mu(M') = W'$. By definition of M', all w in W' have rejected acceptable offers from M' and thus w_0 held engagement xm_0 (ym_0) from some $m_0 \in M'$ where $x, y \in \{w_0, m_0\}$ and $x \neq y$ when receiving the last acceptable plan. Clearly, $m_0 \notin M'$, otherwise m_0 must make another proposal to someone in W' since $\mu_2^M(m_0) \in W'$. Note that $xw_0 \succ_{m_0} \mu^M(m_0) \succ_{m_0} \mu(m_0)$, and $ym_0 \neq \mu(w_0)$ then $\mu(w_0)$ has been

rejected in favor of ym_0 since no more acceptable proposals from M' are received by w_0 after ym_0 has been rejected. Then $ym_0 \succ_{w_0} \mu(w_0)$. Thus (m_0, w_0) blocks μ.
Case 2: If $\mu_2^M(M') \neq \mu_2(M')$, then $\exists w_0 \in \mu_2(M') - \mu_2^M(M')$, s.t. $w_0 = \mu_2(m_0)$ for some $m_0 \in M$. Denote $\mu_2^M(w_0) = m$, then to avoid blocking of μ^M, $\mu^M(w_0) \succ_{w_0} \mu(w_0)$. On the other hand, since $w_0 \notin \mu_2^M(M')$, $m \in M'$ and $\mu^M(m) \succ_m \mu(m)$. Thus (m, w_0) blocks μ. This completes the proof for the lemma.
Now we come to the proof of Theorem 4 (2) using the example of M-proposing P-DAFC. The W-proposing case is exactly the same.
Suppose by way of contradiction that some nonempty set $M_l \subset M$ are strictly better off by misreporting their preferences as P'. Denote μ' as the outcome of M-proposing P-DAFC under (M_1, M_2, W_1, W_2, P') and μ^M as the outcome under(M_1, M_2, W_1, W_2, P). Clearly μ' is individually rational under true preferences since liars all benefit from misstatements. Note that $\mu'(m) \succ_m \mu^M(m), \forall\ m\ \in M_l$. Then we can apply the above lemma to the market (M_1, M_2, W_1, W_2, P) since $M_l \subset M'$. Then $\exists (m, w)$ that blocks μ' unde P such that $\mu^M(m) \succ_m \mu'(m)$. This implies $m \notin M_l$ and thus $P(m) = P'(m)$. Also, $P(w) = P'(w)$. Thus (m, w) blocks μ' unde P', contradicting with the stability of μ' under P'. This completes the proof. □

Proof of Theorem 5: (1). Suppose by way of contradiction that μ is DSFC but pareto dominated by another matching μ'. WLOG, assume that $\exists m \in M$ such that $\mu'(m) \succ_m \mu(m) \succeq_m mm$. Then $\exists w \in W$ s.t. $\mu_2'(m) = w$. Since μ' pareto dominates μ, and $\mu'(m) \neq \mu(w)$, we know that $\mu'(w) \neq \mu(w)$ and $\mu'(w) \succeq_w \mu(w)$. By strictness, $\mu'(w) \succ_w \mu(w)$. Thus m and w block μ, which contradicts with its dynamic stability.
(2). With symmetry, we just prove the result for μ^M. Suppose by way of contradiction that μ is indiviudally rational and $\mu \succ_m \mu^M\ \forall m \in M$. With strictness, μ^M is the outcome of the Men-proposing P-DAFC. Denote $W^{\mu^M} = \{w | \mu_2^M(w) \neq w\}$. Firstly, we show that $W^{\mu^M} = W^{\mu}$. On the one hand, if $\exists w \in W^{\mu} \backslash W^{\mu^M}$, then $\exists m \in W$ s.t. $\mu_2(m) = w$ and $\mu(m) \succ_m \mu^M(m)$ and $\mu^M(w) = ww$. Since $\mu(w)$ is acceptable for w, (m, w) blocks μ^M, which is a contradiction. Thus $W^{\mu} \subset W^{\mu^M}$. On the other hand, note that $|W^{\mu}| = |M^{\mu}|$, $|W^{\mu^M}| = |M^{\mu^M}|$ and $M^{\mu^M} \subset M^{\mu}$ (as every man is strictly better off and thus matched), thus $|W^{\mu^M}| \leqslant |W^{\mu}|$. Then $W^{\mu^M} = W^{\mu}$. Similarly, we can show that $M^{\mu^M} = M^{\mu}$. Note that everyone in M is strictly better off, $M^{\mu^M} = M^{\mu} = M$, this implies that $|W| \geqslant |M|$ and the P-DAFC will terminate immediately when everyone woman in W^{μ^M} receives any acceptable plan. Secondly, denote w_0 as the last one to receive an acceptable plan in M-proposing P-DAFC, $w_0 \in W^{\mu^M} = W^{\mu}$, then $\exists m_0 \in M$ s.t. $\mu_2(w_0) = m_0$. By definition, m_0 must have proposed to w_0 in the M-DAFC algorithm but got rejected, which is a contradiction. This completes the proof. □

Proof of Theorem 6: Consider μ and μ' are two dynamically stable matchings with full commitment. By symmetry, we only prove that $\lambda \equiv \mu \vee_M \mu'$ is dynamically stable with full commitment by the following steps.

Step 1: λ is a matching. It suffices to show that $\forall m \in M, \forall w \in W$, $\lambda(m) = yw \Longleftrightarrow \lambda(w) = zm$, where $y, z \in \{m, w\}$ and they reflect each other.[24]

"$\Longrightarrow$" Suppose that $\lambda(m) = yw$ where $y \in \{m, w\}$. WLOG, assume $\mu(m) = \lambda(m) = yw$ and suppose by way of contradiction that $\lambda(w) \neq \mu(w) = zm$. Then $\lambda(w) = \mu'(w) \neq \mu(w)$. By the definition of λ and the strictness of preferences, we have $zm = \mu(w) \succ_w mu'(w)$. On the other hand, since $\mu'(w) \neq \mu(w)$ and μ, μ' are matchings, $\mu'(m) \neq \mu(m) = yw$. Again the definition of λ implies that $\mu(m) = yw \succ_m \mu'(m)$. Now we have (m, w) blocks μ' with m being matched to yw. This is a contradiction and thus $\lambda(w) = \mu(w) = zm$. Necessity holds.

"$\Longleftarrow$" Define $M' \equiv \{m \in M : \exists w \in W, y \in \{w, m\}, s.t. \lambda(m) = yw\} = \{m \in M : \exists w \in W, y \in \{w, m\}, s.t. \mu(m) = yw,\ OR\ \mu'(m) = yw\}$, $W' \equiv \{w \in W : \exists m \in M, z \in \{w, m\}, s.t. \lambda(w) = zm\} = \{w \in W : \exists m_1 \in M, z_1 \in \{w, m_1\}, s.t. \mu(w) = z_1 m_1,\ AND\ \exists m_2 \in M, z_2 \in \{w, m_2\}, s.t. \mu(w) = z_2 m_2\}$.
From the analysis in "$\Longrightarrow$", we know that $\lambda(m) = yw \Longrightarrow \lambda(w) = zm$.
Define $W'' \equiv \{w \in W : \exists m \in M' \subset M, y \in \{w, m\}, s.t. \lambda(m) = yw\}$, then we have $W'' \subset W'$ and $|W''| \leqslant |W'| \cdots ①$.
Now we prove that $|W''| = |M'| \cdots ②$. On the one hand, $\forall w_1 \neq w_2 \in W''$, there must $\exists m_1 \neq m_2 \in M'$ and $y_1 \in \{w, m_2\}, y_2 \in \{w, m_2\}$ s.t $\lambda(m_1) = y_1 w_1$ and $\lambda(m_2) = y_2 w_2$, otherwise $\lambda(m) = y_1 w_1 \neq y_2 w_2 = \lambda(m)$ and this is a contradiction. Thus, $|W''| \leqslant |M'|$.
On the other hand, by definition of M' and W'', $\forall m \in M'$, there exists $w \in W''$ s.t. $\exists y \in \{m, w\}$ and $\lambda(m) = yw$. Suppose that $|W''| < |M'|$, then $\exists m \neq m' \in M'$, $w \in W''$, $y_1 \in \{m, w\}$ and $y_2 \in \{m', w\}$ s.t. $\lambda(m) = y_1 w$ and $\lambda(m') = y_2 w$. From "$\Longrightarrow$", we derive $\lambda(w) = z_1 m$ and $\lambda(w) = z_2 m'$. Again a contradiction since $m \neq m'$.
Also, the fact that μ is a matching implies that $|\mu(W')| = |W'| \cdots ③$.
Notice that $\forall w' \in W', \exists m \in M$ and $z \in \{m, w'\}$ s.t. $\mu(w') = zm$. Since μ is a matching, $\mu(m) = yw'$ where y is the reflection of z and then $m \in M'$. Thus the nature of a matching guarantees that different $w \in W'$ corresponds to different $m \in M'$ and $|\mu(W')| \leqslant |M'| \cdots ④$.
By ①②③④, $|W'| \geqslant |W''| = |M'| \geqslant |\mu(W')| = |W'| \Longrightarrow |W'| = |W''| \xrightarrow{W'' \subset W'} W'' = W' \cdots ⑤$.
Let $w \in W$,

- If $w \notin W'$, then by definition of W', $\lambda(w) = ww$;
- If $w \in W'$, then $\exists m \in M, \exists z \in \{m, w\}$, s.t. $\lambda(w) = zm$. From ⑤, $\exists m' \in M', y \in \{m', w\}$ s.t. $\lambda(m') = yw$. By necessity, $\lambda(w) = z'm'$ where z' reflects y. Because $\lambda(w) = zm$, we have $z' = z, m' = m$ and $\lambda(m) = yw$. That is, $\lambda(w) = zm \Longrightarrow \lambda(m) = yw$.

[24] That is, $y = m \Longleftrightarrow z = w$ and $y = w \Longleftrightarrow z = m$.

This completes the proof that μ is a matching.

Step 2: λ is dynamically stable with full commitment. Firstly, since μ and μ' are individually rational and $\lambda(x) = \mu(x)$ or $\mu'(x)$, λ is individually rational.

Secondly, suppose by way of contradiction that (m,w) blocks λ with m being matched to yw for some $y \in \{m,w\}$. Let z be the reflection of y, then the blocking suggests that $zm \succ_w \lambda(w)$ and $yw \succ_m \lambda(m) \Longrightarrow$ "$yw \succ_m \mu(m)$ and $yw \succ_m \mu'(m)$", "$zm \succ_w \mu(w)$ or $zm \succ_w \mu'(w)$". In either case, (m,w) must block μ or μ', which is a contradiction.

Thus, λ is a dynamically stable matching with full commitment. □

Proof of Theorem 7

Step 1: The algorithm will terminate with a two-period matching μ. Firstly, the algorithm will continue if and only if when there are new proposals rejected in the current round, but the possible number of rejections is bounded from above by $|M| \times |W|$. Thus the algorithm will terminate. Secondly, note that both $\mu_1 = \mu_1^I$ and $\mu_2 = \bar{\mu}_2$ are determined via some version of DA algorithm, it is easy to see they are spot matchings and μ is a two-period matching.

Step 2: μ is individually rational. On the one hand, $\mu(i)$ must be acceptable to i as in every round, no unacceptable offers are made or held; On the other hand, for $\mu_1(i) \neq i$, $i \in T_1 \cup T_3$. If $i \in T_3$, then $\mu(i) = (\mu_1(i), i) \succsim_i (\mu_1(i), \mu_1(i))$. If $i \in T_1$, then $\mu^I(i) = (\mu_1(i), \mu_1(i))$. Note that in Stage 2, i must be weakly better off than μ^I, thus $\mu(i) \succsim_i (\mu_1(i), \mu_1(i))$. This show that IR is satisfied.

Step 3: μ is not period-1 blocked by any pair involving T_3. Note that in the algorithm, type-3 agents are indifferent between μ_I and μ. For any $m \in T_3$ and (m,w) period-1 blocks μ, $w \notin T_2$ since m prefers to be unmatched in period 2 and w prefers to be unmatched in period 1. Also, $w \notin T_1$ since by rankability $mm \succ_w mw \succ_w \mu(w)$ and $wm \succ_m mm \succ_m ww$, which is excluded by Condition (iii) in the definition of period-1 blocking. Thus, $w \in T_3$, but in this case m must have proposed to w in Stage 1 algorithm but got rejected, which implies that $\mu(w) \succ_w mw$. Contradiction! We can similarly prove the case for $w \in T_2$

Step 4: μ is not period-1 blocked by any pair involving $T_1 \cap \{i : \mu_1(i) \neq i\}$. WLOG, suppose by contradiction that $m \in T_1 \cap \{i : \mu_1(i) \neq i\}$ and (m,w) period-1 blocks μ. Firstly, with weak dynamic stability, we do not allow m to form a blocking pair with some $w \in T_2$. We have also shown that $w \notin T_3$ by in Step 3. Then, if $w \in T_1$, by rankability, we know that $ww \succ_m \mu(m)$ and $mm \succ_w \mu(w)$. This means thath m has proposed to w in Stage 1 algorithm but been rejected in favor of a better plan and thus $\mu(w) \succ_w mm$. Contradiction!

Step 5: If μ is not period-2 blocked by any coalition, then it is not period-1 blocked by any pair of agents among $T_1 \cup T_3$ or among $T_2 \cup T_3$ or among $T_2 \cup (T_1 \cap \{i : \mu_1(i) = i\})$. We have shown that μ is not period-1 blocked by any pair involving agents in T_3 and $T_1 \cap \{i : \mu_1(i) \neq i\}$. Then the only possible period-1 blocking pairs lie in $T_2 \cup (T_1 \cap \{i : \mu_1(i) = i\})$. Suppose (m,w) is one of such pairs, then $\mu_1(m) = m, \mu_1(w) = w$. Denote $S = \{m,w\}$, $\overline{\mu}'_2(m) = w$, then S will period-2 block μ via $\overline{\mu}'_2$. Actually, such a blocking pair will be removed by the Stage 2 algorithm as m must have proposed to w but got rejected. Thus, now we only need to show that μ is not period-2 blocked by any coalition.

Step 6: μ is not period-2 blocked by any coalition. We firstly prove Lemma 1, which characterizes possible proposals in Stage 2.
Lemma 1: (i). If $m \in D^M$, then $E^m \subset D^P \cup D^W$; (ii). If $m \in D^P$, then $E^m \backslash \mu_2^I(m) \subset D^W$

Proof of Lemma 1: The first part of the lemma is evident since $W \subset D^P \cup D^W$. For (ii), if $m \in D^P, \mu_2^I(m) = w \in D^P$ and $\exists\, w_1 \in D^P \cap E^m$, $w_1 \neq w$. Then $ww_1 \succ_m ww = \mu^I(m)$ and $(\mu_1^I(w_1), m) \succ_m \mu^I(w_1)$. By rankability, $w_1w_1 \succ_m \mu^I(m)$ and $mm \succ_{w_1} \mu^I(w_1)$, which means that (m, w_1) period-1 blocks μ and leads to a contradiction with Step 5. Thus, $D^P \cap E^m = \emptyset$ and thus $E^m \backslash \mu_2^I(m) \subset D^W$.

Lemma 2: If $\mu^I(m') = w'w'$ and w' received some proposal in the Stage 2 algorithm, then $\mu(m') = w'w'$ if and only if m' has been rejected by $E^m \backslash \{w'\}$.

Proof of Lemma 2: Firstly, for $m' \in D^P$, $E^{m'} \backslash \{w'\} \subset D^W$. To see this, we know that $T_3 \cap E^{m'} = \emptyset$ since type-3 agents prefer to be unmatched in period 2. Also, if there exists $w \neq w'$ and $w \in E^{m'} \cap T_1$, then $w'w \succ_{m'} w'w'$ and $(\mu_1^I(w), m') \succ_w \mu^I(w)$. By rankability, $(ww \succ_{m'} \mu^I(m'))$ and $m'm' \succ_w \mu^I(w)$. Then m' must have proposed tp w in Stage 1 algorithm and w has rejected m' in favor of a better plan, which is a contradiction. Thus, $E^{m'} \backslash \{w'\} \subset D^W$.
Once w' has received a proposal, m' becomes triggered and begins to propose to agents in $E^{m'}$. Then m' will offer to w' and get accepted once again if and only if he is rejected by all $w \in E^{m'} \cap D^W = E^m \backslash \{w'\}$. We should also notice that $w \in D^W$ is becoming weakly better off as the Stage 2 algorithm proceeds and thus m' will still propose to w' even when w' be retriggered afterwards.

With the above preparations, we can prove the nonexistence of period-2 blocking set. Suppose by contradiction that S period-2 blocks μ. It is clear that there must exist some $m \in S$ to be strictly better off, otherwise $\overline{\mu}_2(i) = \mu_2(i)$ for any $i \in S$. There are three cases.

(1) If $m \in D^M$, then $\overline{\mu}_2(m) = w \in D^W \cup D^P$. Thus, $w \in S$ and is strictly better off with m: $(\mu_1^I(w), m) \succ_w \mu(w)$. Also, m must have proposed to w in Stage 2 algorithm but got rejected, not because that w has received a better plan. The only possible case is that $w \in D^P$ has been triggered but $\mu_2^I(w) = m_1$ finally proposed to w in Stage 2 and $\mu(w) = m_1 m_1$. Thus, $m_1 \in S$ is strictly better off in $\overline{\mu}_2$. Since $E^{m_1} \backslash \{w\} \subset D^W$, $\overline{\mu}_2(m_1) = w_1 \in D^W$. This suggests that m_1 has proposed to w_1 in Stage 2 but got rejected in favor of a better proposal, and w_1 is strictly worse off in the blocking matching $\overline{\mu}_2$. This contradicts with the definition of period-2 blocking coalition.

(2) If $m \in D^P \cap S$ and $\mu_2^I(m) = w$ has received a proposal in Stage 2, then m is triggered in Stage 2. Since m is strictly better off in the blocking set, $\overline{\mu}_2(m) = w_1 \in D^W$. Thus m must have proposed to w_1 but got rejected in favor of some better partner by w_1. This means that w_1 is worse off in the blocking matching $\overline{\mu}_2$. Contradiction.

(3) If $m \in D^P \cap S$ and $\mu_2^I(m) = w$ has not received a proposal in Stage 2, then $\mu(m) = ww$ and $w \in S$, $\overline{\mu}_2(w) \neq m$. Thus, $\overline{\mu}_2(w) = m_1 \in D^M$. Then $m_1 \in S$ and thus $m_1 w \succ_{m_1} \mu(m_1)$, which suggests that m_1 has proposed to w in Stage 2, which contradicts with the assumption.

Thus there is no period-2 blocking coalition. This completes the proof. □

Proof of Theorem 8

Step 1: The algorithm will terminate with a two-period matching μ. Firstly, for Stage 1 and Stage 2, the algorithm will continue if and only if when there are new proposals rejected in the current round, but the possible number of rejections is bounded from above by $|M| \times |W|$. For Stage 3, the algorithm will continue if and only if when there are new cycles formed, which means new agents are removed in the current round. Again, this is bounded by the amound of agents on the market. Thus the algorithm will terminate. Secondly, note that both $\mu_1 = \mu_1^I$ and μ_2^{II} are determined via some version of DA algorithm, and $\mu_2 = \mu_2^{III}$ is determined via TTC algorithm for $F^{un} \cup F^{off}$ and $\mu_2^{III}(F^{un} \cup F^{off}) = F^{un} \cup F^{off}$, it is easy to see they are spot matchings and μ is a two-period matching.

Step 2: μ is individually rational. We have shown in the observations that $\forall i \in M \cup W$, $\mu(i) \succ_i \mu^{II}(i) \succ_i \mu^I(i)$. Thus, it suffices to prove that μ^I is individually rational. On the one hand, $\mu(i)$ must be acceptable to i as in every round, no unacceptable offers are made or held; On the other hand, for $\mu_1(i) \neq i$, $i \in T_1 \cup T_3$. If $i \in T_3$, then $\mu(i) = (\mu_1(i), i) \succsim_i (\mu_1(i), \mu_1(i))$. If $i \in T_1$, then $\mu^I(i) = (\mu_1(i), \mu_1(i))$. Note that in Stage 2, i must be weakly better off than μ^I, thus $\mu(i) \succsim_i (\mu_1(i), \mu_1(i))$. This show that IR is satisfied.

Step 3: μ is not period-1 blocked by any pair. Similarly, it suffices to show the result for μ^I. Suppose by contradiction that (m,w) period-1 blocks μ^I.

(1) If $m \in T_3$, then $w \notin T_2$ as the only mutually acceptable matching among them is being unmatched. Also, $w \notin T_1$, since if $mw \succ_w \mu^I(w)$ and $w \in T_1$, by rankability, we have $mm \succ_w mw \succ_w \mu^I(w)$ and then (m,w) cannot block μ. Thus $w \in T_3$, but in this case, m must have proposed to w in Stage 1 but got rejected by another better plan, which means $\mu^I(w) succ_w mw$, again a contradiction.

(2) If $m \in T_2 \cup T_1$, then similarly $w \in T_1 \cup T_2$. In any situation, m must have proposed to w with the most mutually beneficial plan but got rejected in favor of a bette one, which means w ends up with a more preferred partnership plan in μ^I than being matched with m. Thus (m,w) cannot period-1 block μ^I.

Step 4: Characterization of E^m. We prove the following lemma to see what kind of proposals are valid in the algorithm.
Lemma 1: (i). If $m \in D^M$, then $E^M \subset D^{P1}$; (ii). If $m \in D^{P1}$, then $E^m \backslash \{\mu_1^I(m)\} \subset D^{P2} \cup D^W$; (iii). If $m \in D^{P2}$, then $E^m \backslash \{\mu_2^I(m)\} \subset D^{P1}$.

Proof of Lemma 1: (i). For $m \in D^M$, $\mu^I(m) = mm$. Suppose that there exists $w \in E^m - D^{P1}$, then $\mu_1^I(w) = w$ and $mw \succ_m mm = \mu^I(m)$, $wm \succ_w \mu^I(w)$. Thus (m,w) period-1 blocks μ^I, which is a contradiction. So $E^m - D^{P1} = \emptyset$ and $E^m \subset D^{P1}$.
(ii). For $m \in D^{P1}$, $\mu^I(m) = ww$. Suppose that there exists $w_1 \in (E^m \backslash \{w\} - (D^{P2} \cup D^W))$, then $w_1 \in D^{P1}$ and $\exists\, m_1$ s.t. $\mu^I(w_1) = m_1 m_1$ and $m_1 m \succ_{w_1} m_1 m_1 = \mu^I(w_1)$, $ww_1 \succ_m ww = \mu^I(m)$. By rankability, $mm \succ_{w_1} m_1 m_1 = \mu^I(w_1)$, $w_1 w_1 \succ_m ww = \mu^I(m)$ and thus (m,w) period-1 blocks μ^I. Again a contradiction.
(iii). For $m \in D^{P2}$, $\mu^I(m) = mw$ for some w. Suppose that there exists $w_1 \in (E^m \backslash \{w\} - D^{P1})$, then $\mu_1^I(w_1) = w_1$. Since $w_1 \neq w$ and $w_1 \in E^m$, $mw_1 \succ_m mw = \mu^I(m)$ and $w_1 m \succ_{w_1} \mu^I(w_1)$, which means (m,w_1) period-1 blocks μ^I, contradicting with Step 3. Thus, $E^m \backslash \{\mu_2^I(m)\} \subset D^{P1}$.

Step 5: μ is not period-2 blocked by any S with $\overline{\mu}_2$ such that $\exists i \in S - (F^{un} \cup F^{off})$ strictly better off in $\overline{\mu}_2$.
Suppose by contradiction that S period-2 blocks μ and $i \in S - (F^{un} \cup F^{off})$ is strictly better off in the blocking coalition. Then $i \in D^M \cup D^W \cup F^{on}$.

(1) If $i \in D^M$, denoted as m_0^2, then $\overline{\mu}_2(m_0^2) = w_0 \in S \cap D^{P1}$. Moreover, since m_0^2 have make a proposal to w_0, w_0 cannot be untriggered ($w_0 \notin F^{un}$).

(I) If $w_0 \in F^{on}$, then m_0^2 has proposed to w_0 in Stage 2 algorithm but got rejected in favor of a better agent since w_0 is weakly better off in Stage 2. Thus $\mu(w_0) \succ_{w_0} (\mu_1(w_0), m_0^2)$, a contradiction!

(II) If $w_0 \in F^{off}$, then $\mu_2(w_0) = m_1 \in F^{off} \cap S$. Note that in μ_2, everyone in F^{off} is matched and $\overline{\mu}_2(w_0) \notin F^{off}$. Suppose by contradiction that $\forall i \in S \cap F^{off}$ and $i \neq w_0, \overline{\mu}_2(i) \in F^{off}$. Since $\forall i \in S \cap F^{off}$, $\mu_2(i) \in S \cap F^{off}$, $|S \cap F^{off}| = 2k$ for some positive integer k. Then, $\overline{\mu}_2(S \cap F^{off} - \{w_0\}) = S \cap F^{off} - \{w_0\}$ and there must be some $j \in S \cap F^{off} - w_0$, $\overline{\mu}_2(j) = j$, which means j is worse off in the blocking coalition, contradicting! Thus, $\exists\ i \in S \cap F^{off}$ and $i \neq w_0, \overline{\mu}_2(i) \notin F^{off}$. Moreover, since $M \cap F^{off} = W \cap F^{off} = k$, and $\overline{\mu}_2(w_0) \notin F^{off}$, there exists $m_! \in F^{off}$ such that $\overline{\mu}_2(m_1) = w_1 \notin F^{off}$. Note that $(\mu_1(m_1), w_1) \succ_{m_1} \mu(m_1)$ and m_1 has been triggered in Stage 2, then m_1 must have proposed to w_1 in Stage 2, which means $w_1 \in F^{on} \cup D^W$, which means w_1 would only reject m_1 in favor of a better plan and is weakly better off as the algorithm proceeds. Thus w_1 is worse off in $\overline{\mu}_2$, which is a contradiction!

(2) If $i \in F^{on}$, then $\mu^2(i) \neq \mu^1(i)$ or i and $\mu^2(i) \in S$. Also one of $i, \mu^2(i)$ belongs to M, which is denoted as m_0. $\overline{\mu}_2(m_0) = w_0 \in D^W \cup D^{P1} \cup D^{P2}$.

(I) If $w_0 \in F^{on} \cup D^W$, then m_0 has proposed to w_0 in Stage 2 algorithm but got rejected in favor of a better agent since w_0 is weakly better off in Stage 2. Thus $\mu(w_0) \succ_{w_0} (\mu_1(w_0), m_0)$, a contradiction!

(II) If $w_0 \in F^{un}$ then m_0 has proposed to w_0 in Stage 2 and triggered her, which is a contradiction!

(III) If $w_0 \in F^{off}$, then by the same argument as case (1)-(II), we know that $\exists\ m_1 \in F^{off}$ such that $\overline{\mu}_2(m_1) = w_1 \in F^{on} \cup D^W \cup Fun$, which will lead to a contradiction.

(3) If $i \in D^W$, denoted as w_0, then $\overline{\mu}_2(w_0) = m_1 \in S \cap D^{P1}$. Moreover, m_1 must be untriggered, otherwise m_1 has proposed to w_0 in Stage 2 amd should not have been rejected. This implies that $\mu_2(m_1) = w_1 \in F^{un}$. Then $\overline{\mu}_2(w_1) = m_2 \in S \cap F^{un}$. Suppose by contradiction that $S \backslash \{w_0\} \subset F^{un}$. Again, everyone in F^{un} is matched in period 2 and $|S \cap F^{un}| = 2k$ for some k is a positive integer since $\forall i \in S \cap F^{un}$, $\overline{\mu}_2(i) \in S \cap F^{un}$ and $\overline{\mu}_2(i) \neq i$. However, $\overline{\mu}_2(m_1) = w_0 \notin F^{un}$, thus $\overline{\mu}_2(S \cap F^{un} \backslash \{m_1\}) = S \cap F^{un} \backslash \{m_1\}$. But $|S \cap F^{un} \backslash \{m_1\}| = 2k - 1$ and $(S \cap F^{un} \backslash \{m_1\}) \cup M| = k - 1$. This implies that $\exists\ w \in S \cap F^{un}$ such that $\overline{\mu}_2(w) = w$, which makes w worse off and contradicts with the definition of blocking set. Thus, $S \backslash \{w_0\} \not\subset F^{un}$. Moreover, from the proof above, we know that $\exists w_k \in F^{un}$ such that $\overline{\mu}_2(w_k) \in F^{off} \cup F^{on} \cup D^M$. In any case, w_k would have received a proposal in Stage 2 and been triggered, which contradicts with the assumption that $w_k \in F^{un}$.

Step 6: μ is not period-2 blocked by any coalition.
Firstly, from Step 5, we know that for any period-2 blocking coalition of μ denoted as S via $\overline{\mu}_2$, if $i \in S-(F^{un}\cup F^{off})$, then i is indifferent and thus $\mu_2(i)=\overline{\mu}_2(i)$. Note that $\mu_2(F^{un}\cup F^{off}) = F^{un}\cup F^{off}$ and $\mu_2(S-(F^{un}\cup F^{off})) = \overline{\mu}_2(S-(F^{un}\cup F^{off})) = S-(F^{un}\cup F^{off})$. This implies $\overline{\mu}_2(S\cap(F^{un}\cup F^{off})) = S\cap(F^{un}\cup F^{off})$. Thus $S' \equiv S\cap(F^{un}\cup F^{off})$ should also period-2 block μ via $\overline{\mu}_2$ restricted to $S\cap(F^{un}\cup F^{off})$. In this way, to prove that μ is not period-2 blocked by any coalition, it suffices to show that μ is not period-2 blocked by any coalition in $F^{un}\cup F^{off}$.

For any $m\in F^{off}$, $F^{un}\cap E^m=\emptyset$, since m has proposed (and thus triggered) all agents in $E^m\backslash\mu_2^I(m)$ in Stage 2. Suppose that $S\subset F^{off}\cup F^{un}$ period-2 blocks μ via $\overline{\mu}_2$. Note that all agents in $F^{off}\cup F^{un}$ is matched in μ_2, then by mutual involvement with binding engagements, $S\cap F^{off}=2k_1$ and $S\cap F^{on}=2k_2$ for some k_1,k_2 as two non-negative integers. Suppose that $k_1>0, k_2>0$. If $\exists\ w\in F^{off}$ such that $\overline{\mu}_2(w)\in F^{on}$, then $\exists\ w\in F^{un}$ such that $\overline{\mu}_2(w)=m\in F^{off}$, which is impossible since $w\in F^{un}\cap E^m=\emptyset$. Thus, $\overline{\mu}_2(S\cap F^{off})=S\cap F^{off}$ and $\overline{\mu}_2(S\cap F^{un})=S\cap F^{un}$. Also $\mu_2(S\cap F^{off})=S\cap F^{off}$ and $\mu_2(S\cap F^{un})=S\cap F^{un}$. Thus $S^{un}\equiv S\cap F^{un})$ and $S^{off}\equiv S\cap F^{off})$ period-2 block μ respectively. This implies that we only need to consider blocking sets among F^{off} or F^{un}

For S^{off}, the first observation is that agents removed in the first round (those in G_1) will not join the blocking cycle. To see this, notice that each $m\in G_1$ is matched to his most preferred partner in F^{off} and thus m should be matched to $\mu_2(m)$ if $m\in S^{off}$. Since for any $i\in G_1\cap S^{off}$, $\mu_2(i)\in G_1\cap S^{off}$ by mutual involvement. That is, if $i\in G_1\cap S^{off}$, then $\mu_2(i)=\overline{\mu}_2(2)$, which means that $S'=S^{off}-G_1$ should also period-2 block μ. For simplicity, we assume that $S^{off}\cap G_1=\emptyset$. By induction, we can show that $G_i\cap S^{off}=\emptyset$ for any $i=1,2\dots$ Thus $S^{off}=\emptyset$. Similarly, we can show that $S^{un}=\emptyset$, which means that $S=\emptyset$. Contradition!

Thus, μ is DSTC with binding engagements. This completes the proof. □

Proof of Theorem 9
Step 1: The algorithm will terminate with a two-period matching μ. The same as Proof of Theorem 10, Step 1.
Step 2: μ is individually rational. The same as Proof of Theorem 10, Step 2.
Step 3: μ^I is not period-1 blocked by any pair. The same as Proof of Theorem 11, Step 3.
Step 4: μ is not period-1 blocked by any pair. Firstly, note that $\forall i\in T_3\cup D^P$, $\mu(i)\succ_i \mu^I(i)$, thus there is no blocking pair among $T_3\cup D^P$. Suppose by contradiction that (m,w) period-1 blocks μ.

(1) If $(m,w)\in T_1$, then by rankability, $mm\succ_w \mu(w)$ and $ww\succ_m \mu(m)$.

(I) If $\mu_1(m) \neq m$ and $\mu_1(w) \neq w$, then $m,w \in D^P$, which contradicts as there is no blocking pair among $T_3 \cup D^P$;

(II) If $\mu_1(m) = m, \mu_1(w) \neq w$, then $mm \succ_w \mu^I(w)$. To make sure that (m,w) does not block μ^I, $\mu^I(m) \succ_m ww \succ_m \mu(m) \succsim_m mm$. This implies that $\mu^I_2(m) = w_0 \in T_2 \neq \mu_2(m)$. However, by assumption of SI, $ww \succ_m mw_0 = \mu^I(m)$. Contradiction.

(III) If $\mu_1(m) \neq m, \mu_1(w) = w$, a symmetric case of the above one.

(IV) If $\mu_1(m) = m, \mu_1(w) = w$, with a same argument, we know that $\mu^I_2(m) = w_0 \in T_2$ and / or $\mu^I_2(w) = m_0 \in T_2$. By SI, $mm \succ_w \mu^I(w)$ and $ww \succ_m \mu^I(m)$, which means (m,w) period-1 blocks μ^I, a contradiction!

(2) If $m \in T_1, w \in T_2$ (or similarly, $m \in T_2, w \in T_1$), then $mw \succ_m \mu(m), wm \succ_w \mu(w)$.

(a) If $\mu_1(m) = m$, then (m,w) second-period blocks μ. But note that in Stage 2, m has proposed to w but got rejected in favor of another man, which means $\mu(w) \succ_w wm$. A contradiction!

(b) If $\mu_1(m) \neq m$, then $\exists\ w_1 \in T_1$ such that $\mu^I(m) = w_1w_1$. By SI, $\mu(m) \succsim_m \mu^I(m) = w_1w_1 \succ_m mw$. Thus (m,w) cannot period-1 block μ.

(3) If $m \in T_2, w \in T_2$, then in Stage 2, m must have proposed to w but got rejected in favor of a better proposal from m'. Then $\mu_2(w) \succ_w wm$ and (m,w) cannot period-1 block μ. Contradiction.

Step 5: μ is not period-2 blocked by any coalition S. The same as Proof of Theorem 10, Step 6.

Thus, μ is DSTC. This completes the proof. □

Proof of Theorem 10 (1). Denote μ as a DSTC matching. Suppose by way of contradiction that individually rational μ' Pareto-dominates μ. WLOG, $\exists m \in M$ s.t. $\mu'(m) \succ_m \mu(m) \succeq_m mm$.

I. If $\mu'(m) = mw$ for some $w \in M$,

i. If $\mu'(w) = wm$, then by Pareto dominance, $\mu'(w) = wm \succ_w \mu(w)$ and thus (m,w) period-1 blocks μ via $\mu'(m) = mw$. A contradiction!

ii. If $\mu'(w) = m_1m$ with $m_1 \neq m$, then by Pareto dominance, $\mu'(w) = m_1m \succ_w \mu(w)$. Since μ' is individually rational, $m_1m \succ_w m_1m_1$. Via rankability, $mm \succ_w m_1m_1$ and thus $mm \succ_w m_1m \succ_w \mu(w)$. On the other hand, by rankability, $mw \succ_m \mu(m) \succeq_m mm$ implies $ww \succ_m \mu(m)$. Thus (m,w) period-1 block

II. If $\mu'(m) = ww$ for some $w \in M$, then by Pareto dominance, $\mu'(w) = mm \succ_w \mu(w)$ and thus (m,w) period-1 blocks μ via $\mu'(m) = ww$. A contradiction!

III. If $\mu'(m) = wm$ for some $w \in M$, then $wm \succ_m mm$. Via rankability, $ww \succ_m mm$ and $ww \succ_m wm$.

i. If $\mu'(w) = mw$, then by Pareto dominance, $\mu'(w) = mw \succ_w \mu(w)$. Similarly, we can get $mm \succ_w mw$ by rankability. Thus, (m,w) period-1 blocks μ via $\mu'(m) = ww$. A contradiction!

ii. If $\mu'(w) = mm_1$ with $m_1 \neq m$, then by Pareto dominance, $\mu'(w) = mm_1 \succ_w \mu(w)$. Since μ' is individually rational, $mm_1 \succ_w mm$. Then by rankability, $m_1m_1 \succ_w mm_1 \succ_w \mu(w)$. For m_1, we already have $\mu_2'(m_1) = w$ and $\mu_1'(m_1) \neq w$. If $\mu'(m_1) = m_1w$, then it reduces to the case I. with a contradiction. If otherwise $\mu'(m_1) = w_1w$ for some $w_1 \neq w$, then by individual ratioanlity of μ' and rankability of m_1's preference, $ww \succ_{m_1} w_1w = \mu'(m_1) \succ_{m_1} \mu(m_1)$. Thus (m_1,w) blocks μ, which is a contradiction.

IV. If $\mu'(m) = w_1w_2$ for some $w_1 \neq w_2 \in M$, again by individual ratioanlity of μ' and rankability of m's preference, $w_2w_2 \succ_m w_1w_2 \succ_m \mu(m)$. For w_2, we already know that $\mu_2'(w_2) = m$ and $\mu_1'(w_2) \neq m$. Then with similar arguments for m_1 in case III-ii, it ends up with a contradiction.

Thus, there does not exist an individually rational matching that Pareto dominates some DSTC matching μ. □

(2). To show that T-DA satisfies restricted strategy-proofness, we want to find a simpler but practically identical algorithm for T-DA under rankability, which is strategy-proof in the domain of rankale preferences. WLOG, we study the M-proposing T-DA.

Firstly, in the Stage 1, no man m will make mutually acceptable proposals in the form of wm. To see this, m can only propose such plans among $Y_m^0 \equiv \{wm : w \in W \text{ and } mw \succ_w mm\}$. For $wm \in Y_m^0$, $mw \succ_w mm$, by rankability, $ww \succ_w mm$ and $ww \succ_w mw$. That is, mw is unacceptable for w, which means she will directly reject mw no matter of her status. Thus, the result of Stage 1 algorithm will not be changed if we prevent m from making proposals in Y_m^0. Recall the definition, the Stage 1 algorithm will reduce to the P-DAFC algorithm and thus the interim matching μ^1 is DSFC.

Secondly, we claim that $\mu* = mu^1$, that is everyone will stay with the interim matching in Stage 2. Suppose by contradiction that $\mu^* \neq \mu^1$, that is $\mu_2^* \neq \mu_2^1$. Then $\exists m$ s.t. $\mu_2^*(m) \neq \mu_2^1(m)$, otherwise $\mu_2^* = \mu_2^1$ because of the two-sided nature of matching.

I. If $\mu_2^*(m) = m$, then $\mu_1^1(m) = w \in W$ and $wm \succ_m ww = \mu^1(m)$. By rankability, $mm \succ_m ww$ which is impossible is μ^1 is DSFC.

II. If $\mu_2^*(m) = w \in W$, then $\mu_1^1(m) \neq w$. Denote $\mu_1^*(m) = \mu_1^1(m) = x$, then $xw \succ_m xx$. By rankability, $ww \succ_m xx = \mu^1(m)$. For w, $w \in Q$ and $\mu_2^*(w) = m$. Denote $\mu_1^*(w) = \mu_1^1(w) = y$.

Since w does not leave the Stage 2 algorithm, $ym \succ_w yy$. By rankability, $mm \succ_w yy = \mu^1(w)$. Thus (m,w) blocks μ^1 in the sense of DSFC, which is a contradiction.

Thus, under the assumption of rankability, M-proposing T-DA algorithm (denoted as TDA) produces the same result as M-proposing P-DAFC (denoted as PDA). Because M-proposing P-DAFC is strategy-proof for men, suppose by way of contradiction that M-proposing T-DA is not strategy-proof for men within the domain of rankable preferences. Then, for a market (M,W,P) there exists $m \in M$ with stated preference $P(m)''$ and P'_{-m} as the any profile of preferences for other agents, s.t. $TDA(P(m)'',P'_{-m}) \succ_m TDA(P(m),P'_{-m})$. Then $PDA(P(m)'',P'_{-m}) \succ_m PDA(P(m),P'_{-m})$, which means truthful reporting is not weakly dominant for m under M-proposing P-DAFC. Contradiction! $\square$

Proof of Proposition 1 (1). Given that at most one of $T_2 \cap W$ and $T_3 \cap W$ is nonempty and at most one of $T_2 \cap M$ and $T_3 \cap M$ is nonempty, there are four possible situations: $(T_3 \cap W) \cup (T_2 \cap M) = \emptyset$ or $(T_3 \cap M) \cup (T_2 \cap W) = \emptyset$ or $(T_2 \cap W) \cup (T_2 \cap M) = \emptyset$ or $(T_3 \cap W) \cup (T_3 \cap M) = \emptyset$.
Firstly, We prove for the case $(T_3 \cap W) \cup (T_2 \cap M) = \emptyset$ and the case $(T_2 \cap W) \cup (T_3 \cap M) = \emptyset$ can be shown by symmetry.
If $(T_2 \cap W) \cup (T_3 \cap M) = \emptyset$, that is, $T_2 = T_3 = \emptyset$, then the result directly comes from Theorem 10.
If $(T_2 \cap W) \cup (T_3 \cap M) \neq \emptyset$, then there may arrivals for women and departures for men. Suppose by way of contradiction that μ is DSTC and is Pareto dominated by an individually rational matching μ'.
Firstly, we claim that there exists $m_1 \in M$ being strictly better off in μ'. Otherwise, by strictness of preferences, $\mu(m) = \mu'(m)$ for any $m \in M$. Then by definition of matvhing, $\mu(w) = \mu'(w)$ for any $w \in W$, which means $\mu = \mu'$. Contradiction! Now, we can discuss over the possible types of m_1.

Case 1: $m_1 \in T_1$

I. If $\mu'(m_1) = w_1w_1$ for some $w_1 \in W$, then $w_1 \in T_1$ and by Pareto dominance, $m_1m_1 \succ_{w_1} \mu(w_1)$. Thus, (m_1,w_1) period-1 block μ by $\mu'(w_1) = m_1m_1$;

II. If $\mu'(m_1) = m_1w_1$ for some $w_1 \in W$, by rankability, $w_1w_1 \succ_{m_1} m_1w_1 \succ_{m_1} \mu(m_1)$. Also, $\mu'_1(w_1) \neq m_1$.

   i. If $\mu'(w_1) = w_1m_1$, then (m_1,w_1) period-1 blocks μ via $\mu'(w_1) = w_1m_1$;

   ii. If $\mu'(w_1) = m_2m_1$ with $m_2 \neq m_1$, then $w_1 \in T_1$. By IR $m_2m_1 \succ_{w_1} m_2m_2$. By rankability, $m_1m_1 \succ_{w_1} m_2m_1 \succ_{w_1} m_2m_2$. Thus, (m_1,w_1) period 1 blocks μ via $\mu''(m_1) = w_1w_1$

III. If $\mu'(m_1) = w_1m_1$ for some $w_1 \in W$, then by IR and rankability, we know that $m_1m_1 \succ_{m_1} w_1m_1$, which means μ is blocked by m_1

IV. If $\mu'(m_1) = w_1w_2$ for $w_1 \neq w_2$, then $w_2w_2 \succ_{m_1} w_1w_2 \succ_{m_1} \mu(m_1)$,

i. If $w_2 \in T_1$, then by Pareto dominance, $(\mu'_1(w_2), m_1) \succeq_{w_2} \mu(w_2)$.By IR and rankability, $(m_1,m_1) \succ_{w_2} (\mu'_1(w_2),m_1) \succeq_{w_2} \mu(w_1)$. Thus, (m_1,w_2) blocks μ.

ii. $w_2 \in T_2$, then $\mu'(w_2) = w_2m_1$. For $w_1 \in T_1$, if $\mu'(w_1) = m_1w_1 \succ_{w_1} m_1m_1$, then by rankability, $w_1w_1 \succ_{w_1} m_1w_1$, contradiction with IR. If $\mu'(w_1) = m_1m_2 \succ_{w_1} m_1m_1$, for some $m_2 \neq m_1$, then we know $\mu'(w_1) \succeq_{w_1} \mu(w_1)$ and by rankability, $m_2m_2 \succ_{w_1} \mu(w_1)$. On the other hand, for $m_2 \neq m_1$, since $\mu'_2(m_2) = w_1$, $m_2 \notin T_3$. Thus $m_2 \in T_1$ (since $M \cap T_2 = \emptyset$) and by the same argument as w_1, we get $w_1w_1 \succ_{m_2} \mu(m_2)$. Then μ is blocked by (m_2,w_1).

Case 2: $m_1 \in T_3$, then $\mu'(m_1) = w_1m_1 \succ_{m_1} m_1m_1$ for some $w_1 \in W$ and $w_1m_1 \succ_{m_1} \mu(m_1)$,

I. If $\mu'(w_1) = m_1w_1$, then by Pareto dominance, $m_1w_1 \succ_{w_1} \mu(w_1)$ since $\mu(m_1) \neq w_1m_1$. Thus (m_1,w_1) period 1 blocks μ.

II. If $\mu'(w_1) = m_1m_2$, then $m_2, w_1 \in T_1$. Note that $m_1m_2 \succeq_{w_1} \mu(w_1)$ and $(\mu'_1(m_2), w_1) \succeq_{m_2} \mu(m_2)$, by rankability and IR, $m_2m_2 \succ_{w_1} \mu(w_1)$ and $w_1w_1 \succ_{m_2} \mu(m_2)$. Thus μ is blocked by (m_2,w_1). Contradiction

Secondly, for the case $(T_2 \cap M) \cup (T_2 \cap W) = \emptyset$, there exists $m \in M$ to be strictly better off.

I. If $m \in T_1$, then we claim that $\mu'_2(m) \neq m$. Otherwise, $\mu'(m) = w_0m \succ_m \mu(m) \succeq_m mm$ and by IR, $w_0m \succ_m w_0w_0$. By rankability, $mm \succ_m w_0m$, which is a contradiction. Thus, $\exists w \in W \cap T_1$, s.t. $\mu'_2(m) = w$. By similar argument as above two cases, we get $ww \succ_m \mu(m)$ and $mm \succ_w \mu(w)$ and (m,w) block μ.

II. If $m \in T_3$, then $\exists w \in W$ s.t. $\mu'_1(m) = w$. If further $w \in T_3$, then (m,w) can block μ via $\mu'(m) = wm$. If instead $w \in T_1$, similarly, we can show that $\mu'_2(w) \neq w$. Thus, $\mu'_2(w) = m_1 \in T_1$ and $\mu'_1(w) = m \neq \mu_1(w)$. This implies $\mu'(w) \succ_w \mu(w)$ and we come back to Case I.

Secondly, for the case $(T_2 \cap M) \cup (T_2 \cap W) = \emptyset$, there exists $m \in M$ to be strictly better off.

I. If $m \in T_1$, then we claim that $\mu'_2(m) \neq m$. Otherwise, $\mu'(m) = w_0m \succ_m \mu(m) \succeq_m mm$ and by IR, $w_0m \succ_m w_0w_0$. By rankability, $mm \succ_m w_0m$, which is a contradiction. Thus, $\exists w \in W \cap T_1$, s.t. $\mu'_2(m) = w$. By similar argument as above two cases, we get $ww \succ_m \mu(m)$ and $mm \succ_w \mu(w)$ and (m,w) block μ.

II. If $m \in T_3$, then $\exists w \in W$ s.t. $\mu'_1(m) = w$. If further $w \in T_3$, then (m,w) can block μ via $\mu'(m) = wm$. If instead $w \in T_1$, similarly, we can show that $\mu'_2(w) \neq w$. Thus, $\mu'_2(w) = m_1 \in T_1$ and $\mu'_1(w) = m \neq \mu_1(w)$. This implies $\mu'(w) \succ_w \mu(w)$ and we come back to Case I.

Thus, μ is not Pareto dominated by any individually rational matching.

(2). We only need to consider the following three situations.

I. If $T_2 \cap W \neq \emptyset$ and $T_3 \cap W \neq \emptyset$, then Example 4.2 serve as a counterexample;

II. If $T_2 \cap M \neq \emptyset$ and $T_3 \cap M \neq \emptyset$, then a symmetry of Example 4.2 serve as a counterexample;

III. If $T_2 \cap W$ and $T_2 \cap M$, then see the following example: , $T_1 = \{m_1, m_2, w_1, w_2\}$, $T_2 = \{m_3, m_4, w_3, w_4\}$ and $T_3 = \emptyset$. For preferences,

$m_1 : w_4w_4 \quad w_1w_4 \quad w_2w_4 \quad w_1w_1 \quad w_2w_2 \quad m_1m_1;$ $\qquad m_3 : m_3w_1 \quad m_3m_3$

$m_2 : w_3w_3 \quad w_2w_3 \quad w_1w_3 \quad w_2w_2 \quad w_1w_1 \quad m_2m_2;$ $\qquad m_4 : m_4w_2 \quad m_4m_4$

$w_1 : m_3m_3 \quad m_2m_3 \quad m_2m_2 \quad m_1m_1 \quad w_1w_1;$ $\qquad w_3 : w_3m_2 \quad w_3w_3$

$w_2 : m_4m_4 \quad m_1m_4 \quad m_1m_1 \quad m_2m_2 \quad w_2w_2;$ $\qquad w_4 : w_4m_1 \quad w_4w_4$

The M-proposing refined two-stage TDA algorithm will produce the DSTC matching μ where $\mu(m_1) = w_1w_1$, $\mu(m_2) = w_2w_2$, and the rest of agents are single. However, there is another matching DSTC μ' that Pareto dominates μ and $\mu'(m_1) = w_2w_4 \succ_{m_1} \mu(m_1) = w_1w_1$, $\mu'(m_2) = w_1w_3 \succ_{m_2} \mu(m_2) = w_2w_2$, $\mu'(m_3) = m_3w_1 \succ_{m_3} \mu(m_3) = m_3m_3$, $\mu'(m_4) = m_4w_2 \succ_{m_4} \mu(m_4) = m_4m_4$, $\mu'(w_1) = m_2m_3 \succ_{w_1} \mu(w_1) = m_1m_1$, $\mu'(w_2) = m_1m_4 \succ_{w_2} \mu(w_2) = m_2m_2$, $\mu'(w_3) = w_3m_2 \succ_{w_3} \mu(w_3) = w_3w_3$, $\mu'(w_4) = w_4m_1 \succ_{w_4} \mu(w_4) = w_4w_4$.

This completes the proof. □

Proof of Proposition 2 Denote μ as the result of refined two-stage TDA algorithm. Suppose by way of contradition that μ is Pareto dominated by some individually rational matching μ'. By the same argument in the proof of proposition 1, there exists $m_1 \in M$ being strictly better off in μ'.

1. If there exists $m_1 \in T_2$ such that $\mu'(m_1) = m_1w_1 \succ_{m_1} \mu(m_1)$, then $\mu'_2(w_1) = m_1$.

   (1). If $w_1 \in T_2$, then $\mu'(w_1) = w_1m_1 \neq \mu(w_1)$ and thus (m_1, w_1) blocks μ via $\mu'(m_1) = m_1w_1$;

   (2). If $w_1 \in T_3$, then μ' cannot be individually rational;

   (3). If $w_1 \in T_1$, then to avoid that (m_1, w_1) period 1 blocks μ, $\mu'_1(w_1) \neq w_1$. Also, since $m_1 \in T_2$, $\mu'_1(w_1) \neq m_1$. Thus, $\exists m_2 \neq m_1$, s.t. $\mu'(w_1) = m_2m_1 \succ_{w_1} \mu(w_1)$. Also, by IR, $m_2m_1 \succ_{w_1} m_2m_2$.

   For m_2, $\mu'_1(m_2) = w_1$ and $\mu'_2(m_2) \neq w_1$. If $m_2 \in T_3$, then by assumption $A1$, $m_2w_1 \succ_{w_1} m_2m_2$. By rankability, $w_1w_1 \succ_{w_1} m_2m_2$ and thus $w_1m_1 \succ_{w_1} m_2m_1 \succ_{w_1} \mu(w_1)$, which implies (m_1, w_1) period 1 blocks μ by $\mu''(m_1) = m_1w_1$, a contradiction! This implies that $m_2 \in T_1$.

If $\mu'(m_2) = w_1m_2 \succ_{m_2} m_2m_2$, then rankability implies that $w_1w_1 \succ_{m_2} w_1m_2$, which contradicts with IR of μ'. If instead, $\exists w_2$, s.t. $\mu'(m_2) = w_1w_2 \succ_{m_2} w_1w_1$, then $w_2w_2 \succ_{m_2} w_1w_2 \succ_{m_2} \mu(m_2)$.

For w_2,

I. If $w_2 \in T_1$, denote $\mu'_1(w_2) = x$, similarly we can get $m_2m_2 \succ_{w_2} xm_2 \succ_{w_2} \mu(w_2)$ and thus (m_2, w_2) blocks μ.

II. If $w_2 \in T_2$, then $\mu'(w_2) = w_2m_2$. We already know that $\mu'(m_1) = m_1w_1, \mu'(m_2) = w_1w_2, \mu'(w_1) = m_2m_1, \mu'(w_2) = w_2m_2$. Next we discuss the match of m_2 in μ.

  i. If $\mu_1(m_2) = \mu_1^I(m_2) = m_2$,
  – If $m_2m_2 \succ_{m_2} w_1w_1$, then rankability, $m_2w_2 \succ_{m_2} w_1w_2 = \mu'(m_2) \succ_{m_2} \mu(m_2)$. This means (m_2, w_2) period-1 blocks μ, contradiction.
  – If $w_1w_1 \succ_{m_2} m_2m_2$, then by SI, $w_1w_1 \succ_{m_2} mu^I(m_2)$.

  ii. If $\mu_1(m_2) = w_2 \neq w_1$ or m_2, then $\mu_1(m_2) = w_3$. Notice that $m_2 \in T_1$, $w_3 \in T_1$, then $\mu^1(m_2) = w_3w_3$. As $w_1w_2 \succ_{m_2} w_3w_3$, by $A1$, $w_1w_1 \succ_{m_2} w_3w_3 = \mu^1(m_2)$.

  iii. If $\mu_1(m_2) = w_1 = \mu'_1(m_2)$, then we define $S = \{m_1, m_2, w_1, w_2\}$ and $\overline{\mu}_2(m_1) = w_1$, $\overline{\mu}_2(m_2) = w_2$. Then $(\mu_1(m_1), \overline{\mu}_2(m_1)) = m_1w_1 = \mu'(m_1) \succ_{m_1} \mu(m_1)$, $(\mu_1(m_2), \overline{\mu}_2(m_2)) = w_1w_2 = \mu'(m_2) \succ_{m_2} \mu(m_2)$, $(\mu_1(w_1), \overline{\mu}_2(w_1)) = m_2m_1 = \mu'(w_1) \succ_{w_1} \mu(w_1)$, $(\mu_1(w_2), \overline{\mu}_2(w_2)) = w_2m_2 = \mu'(w_2) \succ_{w_2} \mu(w_2)$. Thus, S period-2 blocks μ with $\overline{\mu}_2$. Contradiction!

By the above argument for m_2, when there is no direct contradiction, we get the intermediate result that $w_1w_1 \succ_{m_2} \mu^1(m_2)$ like in II-i and II-ii. By similarly carrying out the discussion for w_1, either a contradiction arises or $m_2m_2 \succ_{w_1} \mu^1(w_1)$. Thus (m_2, w_1) will block μ^1, again a contradiction! This completes the proof for Case 1. By symmety, we can prove that there does not exist $w \in T_2$ to be strictly better off,

2. If there does not exist $m, w \in T_2$ to be strictly better off, then for any $x \in T_2$, $\mu(x) = \mu'(x)$.

(1). If only type 3 agents are strictly better off, then $\exists m_1 \in T_3, w_1 \in T_3$ s.t. $\mu'(m_1) = w_1m_1$ and both benefit from μ'. Then (m_1, w_1) will block μ.

(2). If $\exists m_1 \in T_1$ such that $\mu'(m_1) \succ_{m_1} \mu(m_1)$, then $\mu'_2(m_1) \neq m_1$.

I. If $\mu'(m_1) = m_1w_1$ for some $w_1 \in W$, then $w_1w_1 \succ_{m_1} \mu(m_1)$. If $w_1 \in T_1$, then also $m_1m_1 \succ_{w_1} \mu(w_1)$ via rankability and IR of μ', and thus (m_1, w_1) blocks μ, contradiction!

If instead $w_1 \in T_2$, then $\mu'(w_1) = \mu(w_1) = w_1m_1$. Accordingly, $\mu_2(m_1) = w_1$. Since

w_1 is strictly better off, $\exists w_2 \in W$ s.t. $\mu(m_1) = w_2w_1$ and $m_1w_1 \succ_{m_1} w_2w_1$. This shows $m_1m_1 \succ_{m_1} w_2w_2$ by rankability. As $\mu^1(m_1) = w_2w_2$, μ^1 is blocked by m_2 solely. Contradiction!

II. If $\mu'(m_1) = w_1w_1$ for some $w_1 \in W$, then $\mu'(w_1) = m_1m_1 \succ_{w_1} \mu(w_1)$ and thus (m_1, w_1) blocks μ.

III. If $\mu'(m_1) = w_2w_1$ for some $w_1 \neq w_2 \in W$, then $w_1w_1 \succ_{m_1} \mu(m_1)$. If $w_1 \in T_1$, similar to the case (2)-I, (m_1, w_1) blocks μ. If $w_1 \in T_2$, then $\mu(w_1) = \mu'(w_1) = w_1m_1$ and $\mu_2(m_1) = w_1$.
– If $w_2 \in T_1$, then we can show that (w_2, m_1) block the interim matching μ^1;
– If $w_2 \in T_3$, then by $A1$, $w_2m_1 \succ_{m_1} \mu^1(m_1)$ and $m_1w_1 \succ_{m_1} \mu^1(m_1)$. Note that $\mu^1(m_1) \neq w_2w_1$, thus at least one of w_1, w_2 are strictly better off at μ' than at μ. Then μ^1 will be blocked for sure.

This completes the proof. $\square$

Proof of Theorem 11: It is easy to show that μ^* is a matching.
Step 1: μ^I is not period-1 blocked by individual or pair of agents. Firstly, in the Stage-1 refined PDA algorithm, s will only propose acceptable plans for him and c will only keep acceptable plans. Thus $\mu(s) \succsim_s ss$ and $\mu(c) \succsim_c cc$.
Secondly, supppose by contradiction that (s, c) period-1 blocks μ. Note that colleges are weakly better off in stage 1. (i) If $ss \succ_c \mu(c)$, $cc \succ_s \mu(s)$ and $cc \succ_s cs$, then $cc \in X_s^4$ and s must have proposed to c with the plan cc but got rejected in Stage 1, which means $\mu(c) \succ_c ss$. Contradiction. (ii). If $cs \succ_c \mu(c)$ and $sc \succ_s \mu(s)$, then again s must have proposed to c with the plan sc but got rejected, thus $\mu(c) \succ_c s$ and (m, w) cannot block μ. (iii) If $sc \succ_c \mu(c)$, $cs \succ_s \mu(s)$ and $cs \succ_s cc$, then $cs \in Y_s^4$ and s must have proposed to c with the plan cs but got rejected in Stage 1, which means $\mu(c) \succ_c sc$. Contradiction.

Step 2: μ^* is not period-1 blocked by individual or pair of agents. We prove it by showing that $\forall\, i \in S \cup C$, $\mu^*(i) \succsim_i \mu^I(i)$.
When $i \in (S \backslash S_1) \cup (C \backslash C_1)$, $\mu^*(i) = \mu^I(i)$ and the claim naturally holds.
When $i \in S_1 \cup C_1$, then $\mu_2^I(i) = i$. For $i \in S_1$, denoted as s, since in period-2, s will only apply for the college c if $(\mu_1^I(s), c) \succ_s (\mu_1^I(s), s)$. Thus $\mu^*(s) \succ_s \mu^I(s) = (\mu_1^I(s), s)$. For $i \in C_1$ denoted as c, again in stage 2 , c will only accepts offers from s if $(\mu_1^I(c), s) \succ_c (\mu_1^I(c), c)$, which means c will also be weakly better off in stage 2. Thus, $\forall\, i \in S \cup C$, $\mu^*(i) \succsim_i \mu^I(i)$.
Then if any individual or pair period-1 blocks μ^*, then it will also period-1 block μ^I, which is a contradiction with Step 1.

Step 3: μ^* is not period-2 blocked by individual. By definition, only students can period-2 block a matching, thus we can only consider agent $s \in S$. By Step 2, it suffices to prove that μ^I is not period-2 blocked by any individual. Suppose by contradiction s period-2 blocks μ^I, then there are two possible cases:

(1) If $(\mu_1^*(s), s) \succ_s \mu^I(s)$, then either $\mu_1^I(s) = s$ or $c \in C$. If $\mu_1^I(s) = s$, then $ss \succ_s ss$, whcih is impossible. If instead $\mu_1^I(s) = c$, then there are two cases:

  (I) If $\mu^I(s) = cc$, then $cs \succ_s \mu^I(s) = cc \succ_s ss$. By $SIC(i)$, $cs \succ_s cc \succsim_s ss \Longrightarrow ss \succ_s cc$, which is a contradiction.

  (II) If $\mu^I(s) = cs$, then again $cs \succ_s \mu^I(s) = cs$, which is a contradiction.

(2) If $(\mu_1^I(s), \mu_1^I(s)) \succ_s \mu^I(s)$, then $\mu_1^I(s) \neq s$ and $\mu^I(s) = cs$ for some $c \in C$. This implies that $cc \succ_s \mu^I(s) = cs \succ_s ss$. For c, since μ^I is not period-1 blocked, $\mu^I(c) = sc \succ_c cc$. By WREP, $ss \succ_c cc$. Suppose that $sc \succ_c ss$, then by $SIC(i)$, $cc \succ_c ss$, which is impossible. Thus $ss \succ_c sc$. Then (s,c) period-1 blocks μ^I with $\mu'(c) = ss$, contradicting.

Step 4: μ^* is not period-2 blocked by coalition. Suppose by contradiction that $A \neq \emptyset$ and $A \subset S \cup C$ period-2 blocks μ^* with $\overline{\mu}_2$.

The first observation is that $A \not\subset S$ and $A \not\subset C$, that is, $\exists\ s \in A$ such that $\overline{\mu}_2(s) \neq s$. Otherwise, if $\forall\ s \in A$, $\overline{\mu}_2(s) = s$, then to avoid that μ^* is period-2 blocked by any $s \in A$, we must have $\mu^*(s) \succsim_s (\mu_1^*(s), s) = (\mu_1^*(s), \overline{\mu}_2(s))$. On the other hand $s \in A$ means that $(\mu_1^*(s), \overline{\mu}_2(s)) \succsim_s \mu^*(s)$, thus by strictness, $\mu^*(s) = (\mu_1^*(s), s)$. This implies that all m in A is indifferent and thus $\exists\ c \in A$, such that $(\mu_1^*(c), \overline{\mu}_2(c)) \succ_c \mu^*(c)$. Since $\overline{\mu}_2(s) = s$ for all $s \in A$, $\overline{\mu}_2(c) = c$. If $\mu_1^*(c) = c$, then $cc \succ_c \mu^*(c)$ and c period-1 blocks μ, which contradicts with Step 2. If $\mu_1^*(c) = s'$, then $s'c \succ_c \mu^*(c) \succsim_c mu^I(c)$, which implies that $\mu_2^I(c) \neq c$. By definition of Stage 1 algorithm, $\mu^I(c) = s's'$. Thus $\mu^I(s') = cc = \mu^*(s')$. Note taht s' will only propose the better plan between cc and cs', thus $cc = \mu^*(s') \succ_{s'} cs'$. Moreover, by mutual involvement, $s' \in A$ and thus $\overline{\mu}_2(s') = s'$, then $cc = \mu^*(s') \succ_{s'} cs' = (\mu_1^I(s'), \overline{\mu}_2(s'))$, contradicting with the definition of blocking set.

Accordingly, $\exists\ s \in A$ such that $\overline{\mu}_2(s) \neq s$. Also, notice that if $\overline{\mu}_2(i) = i$, then by IR of μ^*, $\mu_2^*(i) = i$ and thus i is indifferent in the blocking coalition. This implies that *exists* s, c where $\overline{\mu}_2(s) = c$ and at least one of s, c is strictly better off. Then $\overline{\mu}_2(s) = c \neq \mu_2^*(s)$, and by strictness, both s and c are strictly better off in the blocking set.

Thus, $\exists\ s_0 \in A$ such that $\overline{\mu}_2(s_0) = c_0$ and $(\mu_1^I(s_0), c_0) \succ_{s_0} \mu^*(s_0)$. Also, $c_0 \in A$ and $\mu_2^*(c_0) \neq \mu_2^I(c_0) = s_0$, then $(\mu_1^I(c_0), s_0) \succ_{c_0} \mu^*(c_0)$.

(1) If $\mu_1^*(s_0) = s_0, \mu_1^*(c_0) = c_0$, then (s_0, c_0) period-1 blocks μ^* with $\mu'(s_0) = s_0c_0$, contradiction.

(2) If $\mu_1^*(s_0) = c_0$, then (s_0, c_0) period-1 blocks μ^* with $\mu'(s_0) = c_0c_0$, contradiction.

(3) If $\mu_1^*(s_0) = s_0, \mu_1^*(c_0) = s_1 \neq s_0$, then $s_0c_0 \succ_{s_0} (s_0, \mu_2^*(s_0)) \succsim_{s_0} (s_0, \mu_2^I(s_0)) \succsim_{s_0} s_0s_0$, and $s_1s_0 \succ_{c_0} (s_1, \mu_2^*(c_0)) \succsim_{c_0} (s_1, \mu_2^I(c_0)) \succ_{c_0} c_0c_0$.
For $\mu_2^I(s_0)$, if $\mu_2^I(s_0) = s_0$, then $s_0c_0 \succ_{s_0} s_0s_0 \xRightarrow{SIC(i)} c_0c_0 \succ_{s_0} s_0s_0 = \mu^*(s_0)$; if $\mu_2^I(s_0) = c_1 \neq c_0$, then $s_0c_0 \succ_{s_0} s_0c_1 \xRightarrow{SIC(iii)} c_0c_0 \succ_{s_0} s_0c_1 = \mu^*(s_0)$.
For $\mu_2^I(c_0)$, if $\mu_2^I(c_0) = s_1 \neq s_0$, then $s_1s_0 \succ_{c_0} s_1s_1 \xRightarrow{SIC(i)} s_0s_0 \succ_{c_0} s_1s_1 = \mu^*(c_0)$; if $\mu_2^I(c_0) = c_0$, then $s_1s_0 \succ_{c_0} s_1c_0 \xRightarrow{SIC(ii)} s_0s_0 \succ_{c_0} s_1c_0 = \mu^*(c_0)$.
All cases will lead to the result that (s_0, c_0) period-1 blocks μ^* with $\mu'(s_0) = c_0c_0$. Contradiction.

(4) If $\mu_1^*(s_0) = c_1 \neq c_0, \mu_1^*(c_0) = c_0$, then the case is symmetric with case (3) and the contradiction can occur in the same way.

(5) If $\mu_1^*(s_0) = c_1 \neq c_0, \mu_1^*(c_0) = s_1 \neq s_0$, again, by the argument in case (3) and (4), we can derive that in all possible cases, $s_0s_0 \succ_{c_0} \mu^*(c_0)$ and $c_0c_0 \succ_{s_0} \mu^*(s_0)$, which means that (s_0, c_0) will period-1 blocks μ^*. Contradiction.

From the above four steps, we know that μ^* is DSOCBE and thus the set of DSOCBE matchings is nonempty. □

Proof of Theorem 12: We need to prove that given the limitation on reports, truth telling is the (weakly) dominant strategy for any $s_1 \in S$. Denote the true preference of agent i as $\succsim_i$. Fix a profile of reported preferences of others as $\succsim'_{-s_1}$, denote $H(C, S, R)$ as the PDA-OC mechanism and $\mu = H(C, S, (\succsim_{s_1}, \succsim'_{-s_1}))$ as the DSOCBE matching when s_1 truthfully reports her preference. Suppose by way of contradiction that $\exists\ \hat{\succsim}_{s_1}$ such that $\hat{\mu} = H(C, S, (\hat{\succsim}_{s_1}, \succsim'_{-s_1}))$ and $\hat{\mu}(s_1) \succ_{s_1} \mu(s_1)$.

Firstly, we claim that under the assumptions that reported preferences of students satisfy SIC and those of colleges satisfy SIC and WREP, then $\mu^I = \mu$, that is, the Stage 2 algorithm just has no influence on the final outcome. For ease of illustration, we use the terms like $\succ_i$, but it applies to any profile of reported preferences. Suppose by way of contradiction that $\mu^I \neq \mu$, then $\exists\ s_1 \in S$ such that $\mu^I(s_1) \neq \mu(s_1)$. Note that possible changes are only made to those unmatched in period 2 in μ^I, there are two possible cases for s_1 .

On the one hand, $\mu^I(s_1) = c_1 s_1$ and $\mu(s_1) = c_1 c_2$. If $c_1 = c_2$, then $\mu(s_1) = c_1 c_1 \succ_{s_1} \mu^I(s_1) = c_1 s_1$, $\mu(c_1) = s_1 s_1 \succ_{c_1} \mu^I(c_1) = s_1 c_1$ and thus (s_1, c_1) period-1 blocks μ^I, which is a contradiction. If $c_1 \neq c_2$, then $c_1 c_2 \succ_{s_1} c_1 s_1$ and by *SIC(ii)*, $c_2 c_2 \succ_{s_1} c_1 s_1 = \mu^I(s_1)$. As for c_2, $\mu_2^I(c_2) = c_2$. If $\mu^I(c_2) = c_2 c_2$, then $\mu(c_2) = c_2 s_1 \succ_{c_2} c_2 c_2 \xRightarrow{SIC(iii)} s_1 s_1 \succ_{c_2} c_2 c_2 = \mu^I(c_2)$ and thus (s_1, c_2) period-1 blocks μ^I via $\mu'(s_1) = c_2 c_2$, which is a contradiction. If instead $\mu^I(c_2) = s_2 c_2$ for some $s_2 \neq s_1$, then $\mu(c_2) = s_2 s_1 \succ_{c_2} s_2 c_2 \xRightarrow{SIC(iii)} s_1 s_1 \succ_{c_2} s_2 c_2 = \mu^I(c_2)$. Again, (s_1, c_2) period-1 blocks μ^I via $\mu'(s_1) = c_2 c_2$, which is a contradiction. Thus, there is no s_1 such that $\mu^I(s_1) = c_1 s_1$ and $\mu(s_1) = c_1 c_2$.
On the other hand, $\mu^I(s_1) = s_1 s_1$ and $\mu(s_1) = s_1 c_1$. By *SIC(iii)*, $c_1 c_1 \succ_{s_1} s_1 s_1 = \mu^I(s_1)$. If $\mu^I(c_1) = c_1 c_1$, then (s_1, c_1) period-1 blocks μ^I with $\mu'(s_1) = s_1 c_1$. If $\mu^I(c_1) = s_2 c_1$ for some $s_2 \neq s_1$, then $s_2 s_1 = \mu(c_1) \succ_{c_1} s_2 c_1 \xRightarrow{SIC(ii)} s_1 s_1 \succ_{c_1} s_2 c_1$. Then (s_1, c_1) also period-1 blocks μ^I with $\mu''(s_1) = c_1 c_1$. Contradiction.
Accordingly, when reported preferences of students satisfy SIC and those of colleges satisfy SIC and WREP, then there are no Stage-2 adjustments and $\mu^I = \mu$.

Secondly, we prove a similar version of Lemma C.1 in the proof of strategyproofness in the full-commitment case.
Lemma C.2 Suppose μ' is individually rational, let S' be the set of men who prefer μ' to μ. If S' is nonempty, then there is a pair (s, c) that block μ' such that $s \in S - S'$ and $c \in \mu_1(S') \cup \mu_1(S')$.
Proof of Lemma C.1
$\forall s \in S'$, $\mu'(s) \succ_s \mu(s)$, which implies that $\mu'(s) \neq ss$. Also, for any profile of stated preferences that satisfy the above assumption, the PDA-OC algorithm produces $\mu = \mu^I$, and thus each agent cannot be matched to two different partners in two periods (except for being unmatched), that is, $\nexists\, s \in S$ such that $\mu(s) = c_1 c_2$ for some $c_1 \neq c_2$, and $\nexists\, c \in C$ such that $\mu(c) = s_1 s_2$ for some $s_1 \neq s_2$. Denote $\phi_\mu(A) = \{\mu_1(i) : i \in A, \mu_1(i) \neq i\} \cup \{\mu_2(i) : i \in A, \mu_1(i) = i\}$ as the set of agents who form a partnership with someone in A under μ. Then $\phi_{\mu'}(S') \subset C$.
Case 1: If $\phi_\mu(S') = \phi_{\mu'}(S') \equiv C'$. Denote c_0 as the last one in C' to receive an acceptable plan from someone in S' in M-proposing PDA-OC Stage 1. Then c_0 will not reject this offer, otherwise the corresponding $s \in S'$ will again make an acceptable offer to some $c \in C'$ as $\phi_\mu(S') = C'$. By definition of S' and IR of μ', all c in C' have rejected acceptable offers from S' and thus c_0 held engagement $(x_1, x_2) = s_0 s_0$ or $c_0 s_0$ or $s_0 c_0$ from some $s_0 \in S'$, which will later be rejected, when receiving the last acceptable plan. Denote (y_1, y_2) as the partnership plan corresponding to (x_1, x_2) from the perspective of s_0, that is, $(y_1, y_2) = c_0 c_0$ if $(x_1, x_2) = s_0 s_0$, $(y_1, y_2) = s_0 c_0$ if $(x_1, x_2) = c_0 s_0$, $(y_1, y_2) = c_0 s_0$ if $(x_1, x_2) = s_0 c_0$. Clearly, $s_0 \notin S'$, otherwise s_0 must make another acceptable proposal to someone in C' since $\mu_2(s_0)$ or $\mu_1(s_0) \in \phi_\mu(S') = C'$. Note that $(y_1, y_2) \succ_{s_0} \mu(s_0) \succsim_{s_0} \mu'(s_0)$, and $(x_1, x_2) \neq \mu'(c_0)$ then $\mu'(c_0)$ has been rejected since

no more acceptable proposals from S' are received by c_0 after (x_1,x_2) has been rejected. Then $(x_1,x_2) \succ_{c_0} \mu'(c_0)$. Notice that $\mu(c_0) = (x_1,x_2)$ if and only if $\mu(s_0) = (y_1,y_2)$. Thus (s_0,c_0) blocks μ'.
Case 2: If $\phi_\mu(S') \neq \phi_{\mu'}(S')$, then $\exists c_0 \in \phi_{\mu'}(S') - \phi_\mu(S')$, s.t. $c_0 = \mu'_2(s_0)$ or $c_0 = \mu'_1(s_0)$ for some $s_0 \in S'$. To avoid blocking of μ, $\mu(c_0) \succ_{c_0} \mu'(c_0)$. Thus $\mu(c_0) \neq c_0c_0$. Also, since $c_0 \notin \phi_\mu(S')$, $\exists s_1 \in S - S'$ such that $\mu(c_0) = s_1s_1$ or s_1c_0 or c_0s_1. On the other hand, since $s_1 \notin S'$ and $\mu(s_1) \neq \mu'(s_1)$,$\mu(s_1) \succ_{s_1} \mu'(s_1)$. Thus (s_1,c_0) blocks μ'. This completes the proof for the lemma.

Now we come to the proof of Theorem 12 using the example of M-proposing PDA-OC. The W-proposing case is exactly the same.
Suppose by way of contradiction that some nonempty set $S_l \subset S$ are strictly better off by misreporting their preferences as R'. Denote μ' as the outcome of M-proposing PDA-OC under (C,S,R') and μ as the outcome under(C,S,R). Clearly μ' is individually rational under true preferences since liars all benefit from misstatements. Note that $\mu'(s) \succ_s \mu(s), \forall\ s\ \in S_l$. Then we can apply the above lemma to the market (C,S,R) since $S_l \subset S'$. Then $\exists(s,c)$ that blocks μ' unde R such that $\mu(s) \succ_s \mu'(s)$. This implies $s \notin S_l$ and thus $R(m) = R'(m)$. Also, $R(w) = R'(w)$. Thus (s,c) blocks μ' unde R', contradicting with the stability of μ' under R'. This completes the proof. □

Proof of Theorem 13: For any DSOCBE matching μ , suppose by contradiction that $\exists\mu'$ that is IR and Pareto dominates μ. Then there must be some $s \in S$ such that $\mu'(s) \succ_s \mu(s)$.

(1) If $\mu'(s) = ss$, then μ is period-1 blocked by s, which contradicts with the stability if μ;

(2) If $\mu'(s) = cc \neq \mu(s)$ for some $c \in C$, then $\mu'(c) = ss \neq \mu(c)$ and thus $ss \succ_c \mu(c)$. Then (s,c) period-1 blocks μ which is impossible;

(3) If $\mu'(s) = sc \succ_s ss$, then by $SSIC-(iii)$, $cc \succ_s sc \succ_s \mu(s)$. For c, c is also strictly better off at μ' and $\mu'_2(c) = s$. If $\mu'(c) = cs$, then (s,c) period-1 blocks μ by $\mu'(s) = sc$, leading to a contradiction. If $\mu'(c) = s's$ for some $s' \neq s$. Since $\mu'(c) = s's \succ_c \mu(c)$, by SBFO, $ss \succ_c \mu(c)$. Thus (s,c) period-1 blocks μ via $\mu''(s) = cc$.

(4) If $\mu'(s) = cs \succ_s ss$, then by IR of μ', $cs \succ_s cc$. By $SSIC-(i)$, cc must be unacceptable. For c, $\mu'(c) \neq \mu(c)$ and $\mu'_1(c) = s \neq \mu'_2(c)$. If $\mu'(c) = sc$, then (s,c) period-1 blocks μ via $\mu'(c) = sc$, which is a contradiction. If $\mu'(c) = ss'$ for some $s' \neq s$, then for s, $\mu'(s) \neq \mu(s)$ and $\mu'_2(s) = c \neq \mu'_1(c)$. If $\mu(s) = sc$, then we come back to the case (3); if instead If $\mu(s) = c'c$, then it can be classified to case (5).

(5) If $\mu'(s) = c'c \succ_s ss$ for some $c' \neq c$, then by IR $c'c \succ_s c'c'$ and by $SSIC-(i)$, $cc \succ_s c'c \succ_s \mu(s)$. For c, either $\mu'(c) = cs$ or $\mu'(c) = s's$ for some $s' \neq s$. If $\mu'(c) = cs \succ_c cc$, then by $SSIC-(iii)$, $ss \succ_c cs \succ_c \mu(c)$. Thus (c,s) period-1 blocks μ. If $\mu'(c) = s's \succ_c cc$, then by SBFO, $ss \succ_c \mu(c)$ and again (c,s) period-1 blocks μ, which is a contradiction.

Thus, $\nexists \mu'$ that is IR and Pareto dominates μ. □

Proof of Theorem 14: Via a similar discussion to the proof of theorem 14, we can show that μ^I is a matching that is not period-1 blocked by any agent or pair of agents in $S_I \cup C$ and $\mu^* \succsim_i \mu^I$ for any $i \in S \cup C$. Moreover, μ^* is not period-2 blocked by any individual or any coalition in $S_I \cup C$. Then the following steps are required to prove that μ^* is WDSOCBE.

Step 1: μ^* is not period-1 blocked by individual or pair involving agents in S_{II}. On the one hand, agents in S^{II} will only make acceptable proposals and thus $\mu^*(s) \succsim_s ss$. Thus individualls in S_{II} cannot period-1 block μ. On the other hand, by definition of WDSOCBE, only period-1 blocking pairs in $C \cup S_I$ are allowed, which is not applicable here.

Step 2: μ^* is not period-2 blocked by individual in S_{II}. Since any $s \in S_{II}$ will not make any proposal in Stage 1 algorithm, $\mu_1^*(s) = s$. Thus period-2 unilateral blocking coincides with the period-1 unilateral blocking, which is excluded by step 1.

Step 3: μ^* is not period-2 blocked by coalition involving agents in S^{II}. Suppose by contradiction that $A \cap S_{II} \neq \emptyset$, and $A \subset S \cup C$ period-2 blocks μ^* with $\overline{\mu}_2$.

The first observation is that $\exists\ s_0 \in S_{II} \cap A$ and $\mu_2^*(s_0) \neq \overline{\mu}_2(s_0)$. Otherwise if for all $s \in S_{II} \cap A$, $\mu_2^*(s) = \overline{\mu}_2(s)$. Then $\overline{\mu}_2$ can induce a matching on $(\overline{\mu}_2(S_{II} \cap A) \cup (S_{II} \cap A))$ and thus can also induce a matching on $(A - \overline{\mu}_2(S_{II} \cap A) \cup (S_{II} \cap A))$. Denote $A' \equiv A - \overline{\mu}_2(S_{II} \cap A) \cup (S_{II} \cap A)$. Note that for any $c \in \overline{\mu}_2(S_{II} \cap A) = \mu_2^*(S_{II} \cap A)$, $\mu_1^*(c) \neq \mu_2^*(c)$ since $\mu_2^*(c) \in S_{II}$. This implies that agents in $\overline{\mu}_2(S_{II} \cap A) \cup (S_{II} \cap A)$ will not be involved in a blocking coalition because of the requirement of mutual involvement. Thus A' will also period-2 block μ^* via $\overline{\mu}_2|_{A'}$.

Accordingly, $\exists\ s_0 \in A \cup S_{II}$ such that $\overline{\mu}_2(s) \neq s$. Also, notice that if $\overline{\mu}_2(i) = i$, then by IR of μ^*, $\mu_2^*(i) = i$ and thus i is indifferent in the blocking coalition. This implies that *exists* c_0 where $\overline{\mu}_2(s_0) = c_0$ and by strictness, both s_0 and c_0 are strictly better off in the blocking set.

If $\mu_2^I(c_0) = c_0$, then $c_0, s_0 \in S_1$, which are active in the Stage 2 algorithm. Then s_0 must have applied to c_0 but got rejected in favor of another student since s_0 is acceptable. This is impossible since $(\mu_1^I(c_0), s_0) \succ_{c_0} \mu(c_0)$.

If $\mu_2^I(c_0) = s_1 \in S_I$, then by definition of the algorithm, either $\mu_2^*(c_0) = s_1$ and $\mu_2^*(c_0) = c_0 s_1$ or $s_1 s_1$. In either case, with the requirement of mutual involvement, $s_1 \in A$. Also, since $\overline{\mu}_2(c_0) = s_0 \neq s_1$, $(\mu_1^*(s_1), \overline{\mu}_2(s_1)) \succ_{s_1} \mu^*(s_1)$. Note that μ^* is IR, $\overline{\mu}_2(s_1) \neq s_1$ or c_0. Thus there exists $c_1 \in C \cap A$ such that $c_1 \neq c_0$ and $\overline{\mu}_2(s_1) = c_1$. This implies that $(\mu_1^*(s_1), c_1) \succ_{s_1} \mu^*(s_1)$ and $(\mu_1^*(c_1), s_1) \succ_{c_1} \mu^*(c_1)$. Since μ^* is IR and by $SIC-(i)$ or $SIC-(ii)$, $c_1 c_1 \succ_{s_1} \mu^*(s_1)$. Similarly, by WREP and *SIC*, $s_1 s_1 \succ_{c_1} \mu^*(c_1)$. Thus (s_1, c_1) will period-1 block μ^*, which is a contradiction.

From the above four steps, we know that μ^* is WDSOCBE and thus the set of WDSOCBE matchings is nonempty. □

Proof of Theorem 15: Notice that in PDAAE, the Stage 2 mechanism is a S-proposing DA algorithm restricted to the set of agents unmatched in period 1 in the interim matching μ^I . Moreover, the only difference in the two markets is that $S_{II} \subset S'_{II}$, which will only change the participants of the Stage 2 algorithm. This implies that μ^I is the same for all $C \cup S_I \cup S_{II}$ and it assigns agents in $S'_{II} - S_{II}$ to themselves. Then, to prove that $\mu' \succ_C \mu$, it suffices to show that in the Stage 2 mechanism, colleges are weakly better off in the period-2 spot market with more arrivals. Thus, we just need to show that in a static market, when there are more students, then the colleges will be weakly better of in the S-proposing DA algorithm. This is actually a proposition that has been proved in Roth and Sotomayor (1990) as Theorem 2.25. □

Proof of Theorem 16: It suffices to prove that the outcome μ^* of PDA-OC mechanism is actually DSOCBE. By a similar argument as the proof for theorem 14, μ^I is not period-1 blocked by any agent or pair of individuals, $\mu^* \succsim_{S \cup C} \mu^I$ and thus μ^I is not blocked by any individual, not period-1 blocked by any pair and not period-2 blocked by any coalition in $S_I \cup C$. Thus we only need to show that μ^* is not period-2 blocked by any coalition involving some individual in S^{II}.

Suppose by contradiction that $A \cap S_{II} \neq \emptyset$ and A period-2 block μ^* with $\overline{\mu}_2$. The first observation is that $\exists\ s_0 \in S_{II} \cap A$ and $\mu_2^*(s_0) \neq \overline{\mu}_2(s_0)$. Otherwise if for all $s \in S_{II} \cap A$, $\mu_2^*(s) = \overline{\mu}_2(s)$. Then $\overline{\mu}_2$ can induce a matching on $(\overline{\mu}_2(S_{II} \cap A) \cup (S_{II} \cap A))$ and thus can also induce a matching on $(A - \overline{\mu}_2(S_{II} \cap A) \cup (S_{II} \cap A))$. Denote $A' \equiv A - \overline{\mu}_2(S_{II} \cap A) \cup (S_{II} \cap A)$. Note that for any $c \in \overline{\mu}_2(S_{II} \cap A) = \mu_2^*(S_{II} \cap A)$, $\mu_1^*(c) \neq \mu_2^*(c)$ since $\mu_2^*(c) \in S_{II}$. This implies that agents in $\overline{\mu}_2(S_{II} \cap A) \cup (S_{II} \cap A)$ will not be involved in a blocking coalition because of the requirement of mutual involvement. Thus A' will also period-2 block μ^* via $\overline{\mu}_2|_{A'}$ and $A' \subset C \cup S_I$, which is a contradiction.
Thus, there exists $s_0 \in S_{II} \cap A$, such that $(s_0, \overline{\mu}_2(s_0)) \succ_{s_0} (s_0, \mu_2^*(s_0)) \succsim_{s_0} (s_0, s_0)$. Then $\overline{\mu}_2(s_0) = c_0 \in C \cap A$.

Moreover, we claim that if $\forall\, i \in S_I \cap C$, $\succ_i$ satisfies SIC and WREP, then $\nexists\, s, c$ such that $\mu^I(s) = cs$. To see why this is true, suppose that $\mu^I(s_0) = c_0 s_0$, then $c_0 s_0 \succ_{s_0} s_0 s_0$ and $s_0 c_0 \succ_{c_0} c_0 c_0$. Via WREP, $c_0 c_0 \succ_{s_0} s_0 s_0$ and $s_0 s_0 \succ_{c_0} c_0 c_0$. Via $SIC-(i)$, $c_0 c_0 \succ_{s_0} c_0 s_0 = \mu^I(s_0)$ and $s_0 s_0 \succ_{c_0} s_0 c_0 = \mu^I(c_0)$. Thus s_0 has proposed to c_0 with $\mu'(c_0) = s_0 s_0$ but got rejected, which is impossible.
Now we begin to discuss the possible values of $\mu^I(s_0)$.

(1) If $\mu^I(s_0) = s_0 s_0$, then $s_0 \in S_1$. If $c_0 \in C_1$, then s_0 must have applied to c_0 but got rejected in stage 2, which is impossible as $(\mu_1^*(c_0), s_0) \succ_{c_0} \mu^*(c_0)$. Thus $c_0 \notin C_1$ and $\mu_2^I(c_0) = s_1 \in S$. Then $\mu^*(c_0) = s_1 s_1$ or $c_0 s_1$. In both cases, $s_1 \in A$, which suggests $\overline{\mu}_2(s_1) = c_1 \neq c_0$ and $(\mu_1^*(s_1), c_1) \succ_{s_1} (\mu_1^*(s_1), c_0)$.

   (I) If $s_1 \in S_I$, then $\mu^I(s_1) = c_0 c_0$ or $s_1 c_0$. In either case, by *SIC*, we can show that $c_1 c_1 \succ_{s_1} \mu^I(s_1)$. Similarly, for c_1, $(\mu_1^*(c_1), s_1) \succ_{c_1} \mu^*(c_1) \succsim_{c_1} \mu^I(c_1) \succsim_{c_1} c_1 c_1$. With the previous claim, we know that $\mu^I(c_1) \neq s' c_1$ for some $s' \in S$. And for other cases $\mu^I(c_1) = c_1 c_1$ or $c_1 s_2$ or $s_2 s_2$ with some $s_2 \neq s_1$, we can all show that $s_1 s_1 \succ_{c_1} \mu^I(c_1)$ via *SIC*. Thus (c_1, s_1) will period-1 block μ^I with $\mu'(c_1) = s_1 s_1$ which is a contradiction.

   (II) If $s_1 \in S_{II}$, then $s_1 \neq s_0$ and $\mu^I(s_1) = s_1 c_0 = \mu^*(s_1)$, $s_1 c_1 \succ_{s_1} s_1 c_0 \succ_{s_1} s_1 s_1$. By definition of μ^I, we know that $\mu^I(c_0) = c_0 s_1$, which implies that $(\mu_1^*(c_0), s_0) = c_0 s_0 \succ_{c_0} \mu^I(c_0) = c_0 s_1$. On the other hand, $s_0 c_0 \succ_{s_0} \mu^I(s_0) = c_0 c_0$. Thus (c_0, s_0) period-1 blocks μ^I with $\mu'(c_0) = c_0 s_0$, leading to a contradiction.

(2) If $\mu^I(s_0) = s_0 c_1$ for some $c_1 \neq c_0$, then $s_0 \notin S_1$ and $s_0 c_0 \succ_{s_0} s_0 c_1 = \mu^I(s_0) = \mu^*(s_0)$.
For c_0, $\mu^I(c_0) \neq s' c_0$ by the previous claim. If $\mu_1^I(c_0) = c_0 = \mu_1^*(c_0)$, then $c_0 s_0 \succ_{c_0} \mu^*(c_0) \succsim_{c_0} \mu^I(c_0)$ since $c_0 \in A$. Thus (c_0, s_0) period-1 blocks μ^I with $\mu'(c_0) = c_0 s_0$. Contradiction.
If $\mu^I(c_0) = s_1 s_1$ for some $s_1 \in S_I$, then $s_1 s_0 \succ_{c_0} s_1 s_1 = \mu^*(c_0) = \mu^I(c_0)$. By mutual involvement, $s_1 \in A$. Thus $\exists\, c_2 \in C$ such that $c_0 c_2 \succ_{s_1} c_0 c_0 \succ_{s_1} s_1 s_1$. By $SIC-(i)$, $c_2 c_2 \succ_{s_1} c_0 c_0 = \mu^I(s_1)$. Similarly, for c_2, $(\mu_1^*(c_2), s_1) \succ_{c_2} \mu^*(c_2) \succsim_{c_2} \mu^I(c_2)$, we can show that $s_1 s_1 \succ_{c_2} \mu^I(c_2)$ for all possible cases with *SIC*. Thus (c_2, s_1) period-1 block μ^I with $\mu'(c_2) = s_1 s_1$, which is a contradiction.

This shows that there is no coalition period-2 blocking μ^*, which completes the proof. □

Proof of Theorem 17: Again, it suffices to show that the outcome of PDA-OC is DSOCBE. Notice that the resuls that μ^I is not period-1 blocked by any individual or pair of agents and $\mu^* \succsim_{S \cup C} \mu^I$ do not depend on any assumptions made about preferences, which will still hold in this matching market. Then we only need to show that μ^* is not period-2 blocked by any individual or coalition of agents.

Step 1: μ^* is not period-2 blocked by any individual. By definition, only $s \in S$ can period-2 block μ^* and there are two cases.

(1) If $(\mu_1^*(s), s) \succ_s \mu^*(s)$ for some $s \in S$, consider the possible values of $\mu_1^*(s)$. If $\mu_1^*(s) = s$, then $ss \succ_s \mu^*(s) \succsim_s ss$, which is impossible. If $\mu^*(s) = cc \in C$, then $cs \succ_s \mu^*(s) = \mu^I(s) = cc \succ_s ss$. By $WSIC-(i)$, $ss \succ_s cc$, which is a contradiction. If $\mu^*(s) = cs \in C$, then $cs \succ_s \mu^*(s) \succsim \mu^I(s) = cs$, again a contradiction.

(2) If $(\mu_1^*(s), \mu_1^*(s)) \succ_s \mu^*(s) \succsim_s ss$ for some $s \in S$, then $\exists\ c \in C$ such that $\mu_1^*(s) = c = \mu_1^I(s)$. Also, $\mu_2^*(s) \neq c$ since $cc \succ_s \mu^*(s)$. This implies that $\mu^I(s) \neq cc$ and thus $\mu^I(s) = cs$. Then $cc \succ_s cs = \mu^I(s) \succ_s ss$. For c, since μ^I is IR, $sc \succ_c cc$. Also, $\succ_c$ satisfies WREP and SIC, then $ss \succ_c sc = \mu^I(c)$. (c,s) period-1 blocks μ^I with $\mu'(c) = ss$, leading to a contradiction.

Step 2: μ^* is not period-2 blocked by any coalition. Suppose by contradiction that A period-2 blocks μ with $\overline{\mu}_2$, then it is easy to show that $\exists\ s_0 \in A$ such that $\overline{\mu}_2(s_0) = c_0 \in A$ and $c_0 \neq \mu_2^*(s_0)$. By definition of A, $(\mu_1^*(s_0), c_0) \succ_{s_0} \mu^*(s_0) \succsim_{s_0} \mu^I(s_0) \succsim_{s_0} s_0s_0$ and $(\mu_1^*(c_0), s_0) \succ_{c_0} \mu^*(c_0) \succsim_{c_0} \mu^I(c_0) \succsim_{c_0} c_0c_0$.

(1) If $\mu_1^*(s_0) = s_0$ and $\mu_1^*(c_0) = c_0$, then (s_0, c_0) period-1 blcoks μ^* with $\mu'(s_0) = s_0c_0$, which is impossible.

(2) If $\mu_1^*(s_0) = c_0$, then again (s_0, c_0) period-1 blcoks μ^* with $\mu'(s_0) = c_0c_0$.

(3) If $\mu_1^*(s_0) = c_2 \neq c_0$ and $\mu_1^*(c_0) = c_0$, then there are two possible values for $\mu_2^I(s_0)$. If $\mu_2^I(s_0) = s_0$, then $c_2c_0 \succ_{s_0} c_2s_0 \succ_{s_0} s_0s_0 \xRightarrow{WSIC(ii)} c_0c_0 \succ_{s_0} c_2s_0 = \mu^I(s_0)$. If $\mu_2^I(s_0) = c_2$, then $c_2c_0 \succ_{s_0} c_2c_2 \succ_{s_0} s_0s_0 \xRightarrow{WSIC(i)} c_0c_0 \succ_{s_0} c_2c_2 = \mu^I(s_0)$. In either case, $c_0c_0 \succ_{s_0} \mu^I(s_0)$. Similarly, for c_0, if $\mu_2^I(c_0) = c_0$, then $c_0s_0 \succ_{c_0} c_0c_0 \xRightarrow{SIC(i)} s_0s_0 \succ_{c_0} c_0c_0 = \mu^I(c_0)$. If instead $\mu_2^I(c_0) = s_2 \neq s_0$, then $c_0s_0 \succ_{c_0} c_0s_2 \xRightarrow{SIC(iii)} s_0s_0 \succ_{c_0} c_0s_2 = \mu^I(c_0)$. In either case, $s_0s_0 \succ_{c_0} \mu^I(c_0)$. Thus (s_0, c_0) period-1 blocks μ^I with $\mu'(s_0) = c_0c_0$, which is a contradiction.

(4) If $\mu_1^*(s_0) = c_2 \neq c_0$ and $\mu_1^*(c_0) = s_2 \neq s_0$, then, since $SIC \Longrightarrow WSIC$, by the same argument as s_0 in Case (3) we can derive that $s_0s_0 \succ_{c_0} \mu^I(c_0)$ and $c_0c_0 \succ_{s_0} \mu^I(s_0)$. Again, (s_0, c_0) period-1 blocks μ^I and leads to a contradiction.

(5) If $\mu_1^*(s_0) = s_0$ and $\mu_1^*(c_0) = s_1 \neq s_0$, then $s_0c_0 \succ_{s_0} (s_0, \mu_2^*(s_0)) \succsim_{s_0} (s_0, \mu_2^I(s_0)) \succ_{s_0} s_0s_0$ and $s_1s_0 \succ_{c_0} (s_1, \mu_2^*(c_0)) \succsim_{c_0} (s_1, \mu_2^I(c_0)) \succ_{c_0} c_0c_0$.
For c_0, if $\mu_2^I(c_0) = c_0$, then $s_1s_0 \succ_{c_0} s_1c_0 \succ_{c_0} c_0c_0$, because $\succ_{c_0}$ exhibits bias toward final outcome, $c_0s_0 \succ_{c_0} s_1c_0 = \mu^I(c_0)$. Thus (s_0, c_0) period-1 blocks μ^I with $\mu'(s_0) = s_0c_0$, which is a contradiction.

If $\mu_2^I(c_0) = s_1 \neq s_0$, then $\mu^I(c_0) = s_1 s_1 = \mu^*(c_0)$. By definition of A, $s_1 \in A$. Also $\overline{\mu}_2(s_1) \neq c_0$ since $\overline{\mu}_2(c_0) = s_0$. As $(c_0, \overline{\mu}_2(s_1)) \succ_{s_1} c_0 c_0 \succ_{s_1} c_0 s_1$, where the last relationship comes from the individual rationality of μ^*. Then there exists $c_1 \neq c_0$ such that $\overline{\mu}_2(s_1) = c_1$ and $c_0 c_1 \succ_{s_1} c_0 c_0 = \mu^*(s_1)$ and $(\mu_1^*(c_1), s_1) \succ_{c_1} \mu^*(c_1)$. Thus we can start to discuss s_1, c_1 just like s_0, c_0. Note that $\mu_1^*(s_1) = c_0$, only cases (2), (3), (4) are possible and all will finally lead to a contradiction.

Thus, μ^* is DSOCBE. □

Proof of Proposition 3: We show that the result of RDA is DSOCBE. To begin with, μ^* is a matching since μ_1^I and μ_2^{II} are both spot matchings.

Step 1: μ^* is not period-1 blocked by any individual. For $s \in S$, notice that $\mu^*(s) \succ_s \mu^I(s)$ since s can always choose to be matched with $\mu_1^I(s)$ or stay unmatched in period 2, which gives him the highest conditional ranking. Thus $\mu^*(s) \succsim_s \mu^I(s) \succsim_s ss$. For $c \in C$, if $\mu_1^I(c) = c$, then similarly we can show that c is weakly better off in μ^* than in μ^I and thus will not period-1 block μ^*. If instead $\mu_1^I(c) = s \in S$, then $ss \succ_c cc$. By assumption of KKT, $\mu^I(c) = (s, \mu_2^*(c)) \succ_c cc$.

Step 2: μ^* is not period-2 blocked by any individual. For any $s \in S$, as is shown in Step 2, since s can always choose to be matched with $\mu_1^I(s)$ or stay unmatched in period 2, $\mu^*(s) \succsim_s (\mu_1^*(s), \mu_1^*(s))$ and $\mu^*(s) \succsim_s (\mu_1^*(s), s)$. Then s cannot period-2 block μ^*.

Step 3: μ^* is not period-1 blocked by any pair. Suppose by contradiction that (s, c) period-1 blocks μ^* with μ'. Since μ^* is not blocked by s, $\mu_1^I(s) \neq c$ if $\mu_1'(s) = c$. If $\mu'(s) = cc$, then s must have proposed to c in period 1 but got rejected, which implies that $\mu_1^I(c) = \mu_1^*(c) = s'$ and $s's' \succ_c ss$. By assumption of KKT, $\mu^*(c) \succ_c ss$, which is a contradiction.

If $\mu'(s) = cs$, then similarly s must have proposed to c in period 1 but got rejected, which implies that $\mu_1^I(c) = \mu_1^*(c) = s'$ and $s's' \succ_c ss$. By assumption of KKT, $\mu^*(c) \succ_c ss$, which is again a contradiction.

If $\mu'(s) = sc$, then $cs \succ_c \mu^*(s) \Longrightarrow \mu_1^I(c) = c$. Thus, if s actually applies to c in Stage 2, then she will not be rejected, which is a contradiction. This suggests that $sc \succ_s \mu^*(s) \succ_s (\mu_1^*(s), c)$. Then $\mu_1^*(s) = c_1 \neq c$ and $sc \succ_s (c_1, \mu_2^*(s)) \succ_s c_1 c_1 = \mu^I(s) \succ_s ss$ or $sc \succ_s (c_1, \mu_2^*(s)) \succ_s c_1 s = \mu^I(s) \succ_s ss$. In either case, $c_1 c \succ_s \mu^*(s)$. Thus s must have applies to c in Stage 2, which leads to a contradiction.

Step 4: μ^* is not period-2 blocked by any coalition. Suppose by contradiction that A period-2 blocks μ^* with $\overline{\mu}_2$. Then there must be some $m \in A \cap M$ such that $\overline{\mu}_2(m) = c \neq \mu_2^*(m)$. This implies that $c \in A$ and s, c prefers each other to their partner in μ_2^*. However, s has proposed to c in period-2 but got rejected, which is impossible.

This completes the proof that μ^* is DSOCBE. □

Proof of Proposition 4: Firstly, denote the first-stage interim matching in the M-proposing three-stage TDA algorithm as μ^I, then we know that $\mu^{TC}(i) \succsim_i \mu^I(i)$ for any $i \in M \cup W$. Also, by rankability, for any agent $i \in T_1$, $ji \succ_i ii \Longrightarrow jj \succ_i ii$. Then in stage 1, $m \in M$ will not propose the plan $\mu'(m) = wm$ for some $w \in T_1$; similarly, $winW \cup T_3$ will regard the proposal $\mu'(m) = wm$ from some $m \in T_1$ as unacceptable. Then implies that $\mu^I(T_3) \subset (T_3)^2$ and $\mu^I(T_1 \cup T_2) \subset (T_1 \cup T_2)^2$. Actually, we can equivalently apply the stage 1 algorithm into two separabte markets: $(M \cap T_1, W \cap T_1, (T_1), R|_{T_1})$ and $(M - T_1, W - T_1, (T_2, T_3), R|_{T_2 \cup T_3})$. Then the stage-1 algorithm in three-stage TDA applied to the latter market is identical to the PDA-FC algorithm applied to the orignal market, as in both situations, only plans with full commitment are proposed and accepted. Thus, $\mu^I|_{T_2 \cup T_3} = \mu^{FC}|_{T_2 \cup T_3}$. Also notice that in stage 1, agents in T_3 will not make or accept unacceptable proposals, which implies that $\mu^I(i) \succsim_i \mu^{FC}(i) = ii$ for some $i \in T_3$. Accordingly, $\mu^I \succsim_{M \cup W} \mu^{FC}$ and then $\mu^{TC} \succsim_{M \cup W} \mu^I \succsim_{M \cup W} \mu^{FC}$, which completes the proof. □

Proof of Proposition 5: Suppose that ϕ is a DSFC spot rule. Consider the economy ε_1 where $M_1 = \{m_0\}$, $M_2 = \{m_0, m_1\}$, $W_1 = \{w_0\}$, $W_2 = \{w_0, w_1\}$ and the preferences are as follows: (the preference symbol $\succ$ omitted):

$\mathbf{m_0}: w_1w_1 \quad m_0w_1 \quad w_0w_0 \quad m_0w_0 \quad m_0m_0;$ $\mathbf{m_1}: m_1w_1 \quad m_1w_0 \quad m_1m_1$

$\mathbf{w_0}: m_0m_0 \quad w_0m_0 \quad m_1m_1 \quad w_0m_1 \quad w_0w_0;$ $\mathbf{w_1}: w_1m_0 \quad w_1m_1 \quad w_1w_1$

Then the unique DSFC matching is μ^1 such that $\mu^1(m_0) = m_0w_1$ and $\mu^1(m_1) = m_1w_0$. This implies that for any two-period matching ε market with the above period 1 market for $M_1 = \{m_0\}$ and $W_1 = \{w_0\}$, $\phi(\varepsilon)_1 = \mu^1_1$.

Now consider a different economy ε_2 where the only difference with ε_1 is that the preference of the w_1 prefernce has changed to:

$\mathbf{w_1}: w_1m_1 \quad w_1m_0 \quad w_1w_1$

Notice that the period-1 spot market has not changed, so that $\phi(\varepsilon_2)_1(m_0) = \mu^1_1(m_0) = m_0$. However, it can be shown that the unique DSFC matching in ε_2 is μ^2 such that $\mu^2(m_0) = w_0w_0$ and $\mu^2(m_1) = m_1w_1$. Thus, $\phi(\varepsilon_2) \neq \mu^2$, which implies that $\phi(\varepsilon_2)$ is not DSFC in the market ε_2. Thus ϕ is not a DSFC spot rule. □

Proof of Proposition 6: Recall that we already know that any DSFC matching is Pareto efficient among matchings with full commitment. Thus we only need to prove that the outcome μ^* of RDA-FC is DSFC and M-proposing RDA-FC is strategyproof for the set of men.

(1). Firstly, by defnition, μ^* is a matching with full commitment.

Secondly, for individual rationality, if $i \in E_1$, then by spot stability of μ_1, $\mu_1(i) \succsim^1_i i \Longrightarrow \mu^*(i) \succsim_i ii$; if $i \notin E_1$, then by spot stability of μ_2, $\mu_2(i) \succsim^2_i i \Longrightarrow \mu^*(i) \succsim_i ii$.

Thirdly, suppose by contradiction that (m,w) blocks μ^* with μ', then there are four possible cases.

(1) If $m \in M_1$ and $w \in W_1$, then by weak impatience, $ww \succ_m mw$ and $mm \succ_w wm$. Since (m,w) can block μ^* with μ', (m,w) can also block $\mu''(m) = ww$. This implies that $m \succ_w^1 \mu_1^*(w)$ and $w \succ_m^1 \mu_1^*(m)$, which means that m have proposed to w in the stage 1 DA algorithm but got rejected, which is impossible.

(2) If $m \in M_2 - M_1$ and $w \in W_2 - W_1$, then $\mu_1'(m) = \mu_1^*(m) = m$ and $\mu_1'(w) = \mu_1^*(w) = w$. This implies that $m,w \notin E_1$ and will participate in the stage 2 algorithm. As $\mu'(m) = mw \succ_m \mu^*(m)$, $\mu'(w) = wm \succ_w \mu^*(w)$, $w \succ_m^2 \mu_2(m)$ and $m \succ_w^2 \mu_2(w)$. Then m should have proposed to w in the stage 2 algorithm but got rejected, which is a contradiction.

(3) If $m \in M_1$ and $w \in W_2 - W_1$, then $\mu'(m) = mw$ and $\mu'(w) = wm$. If $\mu_1^*(m) = m$, then by a similar argument to case 2, we can get a contradiction. If instead $\mu_1^*(m) \neq m$, then $\exists\, w_1 \in W_1$ such that $\mu^*(m) = w_1w_1$. Since μ^* is individually rational, $w_1w_1 \succ_m mm$. Then by strong impatience, $\mu(m) = w_1w_1 \succ_m mw = \mu'(m)$, which leads to a contradition.

(4) If $w \in W_1$ and $m \in M_2 - M_1$, then the proof is symmetric to case 3.

Thus, μ^* is DSTC.

(2) To see why the M-proposing RDA-FC is strategyproof for the set of men, take any $m \in M$ with the reported preference $\succsim_m^P$.

If $m \in M_2 - M_1$, then m will only appear on the period-2 market and will not influence μ_1. Recall that in a static market, DA algorithm is strategy-proof for the proposing side, thus m has no incentives to misreport his preference.

If $m \in M_1$, then m strictly prefer to be matched in period one and only the plans in the form of ww will matter for period-1 matching μ_1. Thus, it is weakly dominant for m to report in the strong impatient way, that is, $\forall w_1, w_2 \in W_1 \cup W_2$, $w_1w_1 \succ_m^P mw_2$, which is consistent with real preference $\succsim_m$. Now we only need to consider the rankings within $A_m \equiv \{ww, w \in W_1 \cup W_2\}$ and $B_m \equiv \{mw, w \in W_1 \cup W_2\}$. Notice that period-1 matching is only influenced by reports in A_m, rather than B_m.

For A_m, in the period-1 spot market with the M-proposing DA algorithm, it is weakly dominant for m to report truthfully in the post preference, that is, $w_1 \succ_m^{1,P} w_2 \Longrightarrow w_1 \succ_m^1 w_2$. Thus $w_1w_1 \succ_m^P w_2w_2 \Longrightarrow w_1w_1 \succ_m w_2w_2$. Thus reports within A_m is truthful, regardless of the reporting in B_m. Similarly, given the period-1 matching μ_1 and the set of agents matched in period-1 E_1, if $m \in E_1$, then reports within B_m just does not matter.If instead $m \notin E_1$, then $\mu_1(m) = m$. In the period-2 spot market with the M-proposing DA algorithm, again there is no incentive for m to misreport his ranking within B_m. Thus, the M-proposing RDA-FC is strategyproof for the set of men. □

Proof of Proposition 7: It suffices to prove that μ^* as the outcome of T-period PDA-FC algorithm is dynamically stable with full commitment. Firstly, the algorithm will terminate since the number of proposals are finite and new proposals have to be rejected to let the algorithm persist. Secondly, the outcome is a matching with full commitment since only proposals with consistent plan or engagement can be made and accepted, and μ^* is only determined by the proposals from one side. Thirdly, μ^* is individually rational since men will only proposal acceptable plans and women will only keep acceptable plans. Fourthly, μ^* is not blocked by any pair of agents. Suppose by way of contradiction that (m,w) blocks μ^* with some matching μ' defined over $\{m,w\}$ that exhibits full commitment, such that $\mu'(m) \succ_m \mu^*(m)$ and $\mu'(w) \succ_w \mu^*(w)$, then m must have proposed to w with the plan $\mu'(m)$ but got rejected in some round because w has received a better proposal (since $\mu'(w)$ must be acceptable for w). Notice that women will be weakly better off as the algorithm proceeds and thus $\mu^*(w) \succ_w \mu'(w)$, which is a contradiction. This completes the proof. □

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