Serial and Parallel Duopoly Competition in Multi-Segment Transportation Routes

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Abstract

Commute routes often involve more than one part, frequently on different sources of transport. For example, a worker takes a bus to the subway station, then proceeds to take the subway to work; a traveler takes a taxi to the train station and then catches a train. In our baseline framework, we model the duopoly competition between two transport providers along a route which can be subdivided into two parts. In the serial structure, the route is divided into two parts, with each firm serving as the sole transport supplier along a subroute in a pricing game. In the parallel structure, the two firms compete with one another along the entire route from starting point to ending point. Commuters care about time and price, taking both into consideration in their travel decisions, and may opt to stay home altogether if the travel conditions are insufficiently attractive. Under the serial structure, traffic flows under duopoly are less than that under a monopoly structure, which are less than the traffic flow under the social optimum and price-free scenario, respectively. In the parallel structure, the rank ordering of traffic flows across scenarios differs solely in that the total duopoly traffic flow exceeds the monopolistic traffic flow. The parallel and serial structures are then compared from the perspective of firms, commuters, and social welfare. The parallel structure always yields higher commuter surplus and higher social surplus than the serial structure, however when route conditions are favorable in terms of commute times, the serial structure yields higher profits for the firms. In the extensions of our model, we consider firms' operating costs, more than two firms and more than two segments, and hybrid market structures along the transportation system. Our results suggest a higher overall social desirability of parallel competition along multi-part transportation routes.

Keywords: duopoly, competition, multiple-part journey, transportation routes

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1 Introduction

Many transportation plans involve more than one segment, rather than completing the travel in a single non-stop journey. When a travel route is naturally divided into segments, how does competition among transportation service suppliers determine the ideal market structure for the segments? Take for example the journey between a major city in one state to a small town in another state. A traveler may take a train or flight from the city of departure to the largest city close to the small town. From there, a bus or car ride can take the traveler to the final destination. Many commuters have a two-part journey to their workplaces each morning. Suburban residents might drive to the nearest commuter rail station, park their car there and take the rail into the city. Urban residents may take a bus followed by the subway to their place of employment.

When there is a natural interim point in the journey, is it better to have competing transportation companies supply transport over the entire route (parallel structure), or to divide the route at the interim point and let each serve as the sole supplier over their particular segment (serial structure)? Furthermore, how does the case of competition between suppliers compare to the analogous case of a monopolistic supplier (over the same routing structure), the ideal transportation arrangement implemented by a social planner, and the case of a price-free scenario? In this paper, we analyze these questions in a model in which commuters care about both time and transport price and can decide whether to travel or not, while two firms engage in price competition with one another under the specified market structure over routes.

Our findings favor the parallel competition structure from a total social welfare perspective, as well as for commuters. For the competing firms, the favorability (profit) result depends on the latency conditions of the route segments. For sufficiently favorable travel times, the serial structure provides higher profit for the firms than the parallel structure. This suggests that only when traffic flows are fast moving enough, would there be a conflict of interest between private transportation firms, and the government and commuters they sell to. However, for the vast instances of unfavorable travel flow efficiencies, the welfare of commuters, firms and social planners are aligned in favor of the parallel duopoly market structure.

We consider several extensions of our baseline model, including incorporation of firms' operating costs, generalization of the model to a larger number of firms competing over the transportation segments, and hybrid market structures over the transportation system such as duopoly-monopoly combinations. We find that our main conclusions about the welfare favorability of parallel structures are robust to these variations in the model specification.

Our paper is related to a number of studies which model competition in transportation systems game theoretically via duopoly models. van der Weijde, Verhoef, and van den Berg (2013)[1] consider the equilibrium dynamics between a two-part public transportation system with fees, and an alternative set of unpriced congestible roads. Comparing the scenarios in which the public transport provider is a monopolist versus a serial duopoly, they show that fares under the two market structures can be different, and in some cases, the monopoly fare can be higher than the duopolistic fare. van den Berg (2013)[2] considers the market structure of two transportation infrastructures which must be used in sequence by passengers. Four market structures/timings are considered: Monopoly; Duopoly with simultaneous setting of fees and capacities; Duopoly with sequential setting of capacities followed by fees; and a Stackelberg duopoly in which a leader sets capacity first, and both leader and follower set fees subsequently. The analysis finds that capacity decisions can be influential in the outcomes and social benefits of duopoly compared to monopoly. Our model shares with these previous studies in examining two-part transportation systems, but differs in the following aspects: In contrast to van der Weijde et al (2013)[1], all routes in our model are congestible and priced. While comparing with a monopolistic provider in each case, our main focus is on the comparison of the two types of duopoly, parallel and serial. Finally, in our framework, commuters can elect to not travel if the time and price they face is sufficiently unattractive, whereas their setup focuses on inelastic demand but allows for commuters' departure timing choice. Compared to van den Berg (2013)[2], our analysis assumes Bertrand competition, and focuses on the comparison between serial and parallel duopolies in a pricing game, rather than analyzing the timing of capacity and price decisions by the firms.

More generally, our modeling approach follows the literature on game theoretic analysis of transport providers networks under passenger flows, as in Mazalov and Melnik (2016)[3] and Lien, Mazalov, Melnik and Zheng (2016)[4]. Kuang, Mazalov, Tang and Zheng (2018)[5] examine the transportation network competition problem with externalities.

Our study also bears relation to the literature on multi-modal transportation. Xia and Zhang (2016)[6] analyze the vertical differentiation between high-speed rail and air transport. They find the conditions and price consequences of cooperation and competition between the two popular transport modes in a hub-and-spoke network. Yang, Ban and Mitchell (2017)[7] model emergency evacuation procedures using multimodal transportation network while allowing for capacities, congestion and varying cooperation of evacuees. Almur, Yaman and Kara (2012)[8] consider the multi-modal hub location problem via air transport hubs and ground transport hubs, and conduct cost sensitivity analysis on the locations of the hubs in Turkey. Similarly, Zhang, Yang, Wu and Wang (2014)[9] develop an algorithm to solve the multi-modal transportation network design problem.

When comparing the parallel versus serial structures that we consider in our model, one natural interpretation is the notion of substitutes and complements in transport modes. In the parallel structure, firms are providing substitute modes for consumers, while in the serial structure, firms are providing complementary transport modes. As such, our work is also related to the literature on substitutabilities and complementarity of transport modes. For example, Rus and Socorro[10] (2014) theoretically examine the conditions for investment in a complementary or rival new infrastructure, given an existing infrastructure system. In another theoretical paper, Clark, Jorgensen and Mathisen (2014)[11] study the competition among complementary service providers in a transport chain. On the empirical side, transportation complementarities and substitutability are also of substantial interest. Zhang, Graham and Wong (2018)[12] study the substitutability and complementarity between high-speed rail and air transport in Asia, finding that high speed rail and air travel serve as substitutes on medium and short distance routes, while being complementary on long distance routes in China. Hall, Palsson and Price (2018)[13] conduct an empirical analysis to find that Uber and public transit are on average complements in the United States. Studying the complements and substitutes issue in Beijing, Liu, Jiang, Yang and Zhang (2012)[14] find that public transport and cars are complementary while public transport and bicycles are substitutes.

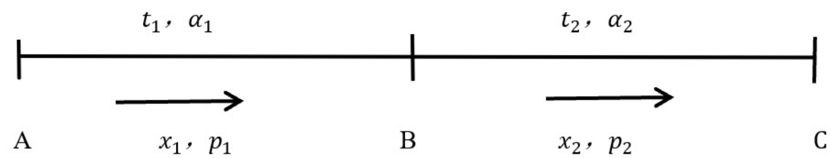
The remainder of the paper is organized as follows: Section 2 begins with the model of serial structure, in which commuters need to travel from A to B and B to C, where each firm is in control of a single segment. We conduct equilibrium analyses for the duopoly case and compare the results with three other alternatives (socially optimal, price-free, and monopolistic scenarios). Section 3 introduces the parallel structure model, in which firms compete on the entire route A to C, and again equilibrium analyses are provided and comparisons across the 4 equilibrium scenarios are drawn. Section 4 introduces the comparison between the serial and parallel structures, including equilibrium traffic flows, prices, profits, consumer and social surpluses. Section 5 extends the model to incorporate firms' operating costs. Section 6 considers the case of more than two firms competing on the transportation structure. Section 7 analyzes the case of hybrid market structures, in other words combinations of duopoly and monopoly over the segments of the transportation network. Finally, Section 8 concludes.

2 Benchmark Model I: Serial Structure

2.1 Settings

Commuters would like to travel from A to C through an intermediate point B. There are two firms providing transport services. Firm 1 is in charge of route AB and sets an entrance fee p_1. Firm 2 is in charge of route BC and sets an entrance fee p_2. All commuters that are willing to travel from A to C must pay $p_1 + p_2$ as total entrance fee. Each firm sets their price strategically but independently of the other firm, in order to maximize their profit, which is the product of travel flow x and their own price, assuming zero operating costs.[2]

[2]For simplicity, in the benchmark models, we assume that firms' operating costs are negligible. In our extension I (Section 5), we allow for non-negligible operating costs and show our main results still hold.



$$x = x_1 = x_2$$

$$u = max\,\{\theta - p_1 - p_2 - f(x), 0\}$$

$$\pi_1 = xp_1 \qquad \pi_2 = xp_2$$

Figure 1: Serial Structure

The value of transport from A to C is a random variable θ that is uniformly distributed within the interval $[0, 1]$. We normalize the size of commuters as a mass of size 1. Commuters take both price and time into consideration in their utility function. For the sake of simplicity, we assume that a commuter is willing to travel if and only if his/her value is larger than (or equal to) the total entrance fee $p_1 + p_2$ plus the travel latency $f(x)$. Therefore, the utility for a commuter is defined as

$$u(\theta, x) = \theta - p_1 - p_2 - f(x) \tag{1}$$

We normalize the utility of the outside option (deciding not to travel, or using an alternative means of travel) as 0.

We assume that the latency of each segment $i = 1, 2$ is linear with respect to total traffic load x,

$$f_i(x) = t_i(1 + \alpha_i x) \tag{2}$$

where t_i indicates the trip time on a completely unoccupied route i. Thus, t_i can also be interpreted as the distance of a route, or representing transportation modes of different speeds. For example, a large t_i could represent a lengthy route, or alternatively a slower transport mode such as cycling. A small t_i could represent either a short route, or a fast transport mode such as high speed rail. Since the commuter's value of transport θ is in the range [0,1], we focus on the reasonable situations in which $t_1 + t_2 \leq 1$. Note that if $t_1 + t_2$ is greater than 1, then no commuters will choose to travel from A to C. α_i is the other latency parameter, which determines how much traffic is slowed due to increased traffic flow. An interpretation of the parameter is the condition of the road, where low α indicates good road conditions and high α indicates poor road conditions. The settings for the benchmark model of serial structure can be illustrated by Figure 1.

In a serial structure, the total latency is hence the summation of the two route segments,

$$f(x) = f_1(x) + f_2(x) = (t_1 + t_2) + (t_1\alpha_1 + t_2\alpha_2)x$$

which still has a linear latency form. We define $t = t_1 + t_2$ and $\alpha = \frac{t_1\alpha_1 + t_2\alpha_2}{t_1 + t_2}$, then we have

$$f(x) = t(1 + \alpha)x \tag{3}$$

where t and α are sufficient for analyzing the serial structure.

In the subsections below, we will first analyze 4 different scenarios (duopoly case, social optimum, price-free, and monopoly), respectively, then conduct comparison of traffic flows among these 4 scenarios, and finally provide results on comparative statics analysis.

2.2 Duopoly Equilibrium

We first derive the demand function when the total entrance fee is p. Assuming the demand (or traffic flow) is x, the equilibrium condition requires that commuters with value greater than or equal to $1 - x$ take the trip while commuters with value less than $1 - x$ choose outside option. Therefore, the demand is given by

$$x = \frac{1 - p - t}{1 + \alpha t} \tag{4}$$

which is solved by $u(1 - x, x) = 0$.

Hence the profits for firm 1 and firm 2 are

$$\Pi_1(p_1) = \frac{1 - p_1 - p_2 - t}{1 + \alpha t} p_1 \tag{5}$$

$$\Pi_2(p_2) = \frac{1 - p_1 - p_2 - t}{1 + \alpha t} p_2 \tag{6}$$

respectively. The profit maximization by each firm results in the following equilibrium prices

$$p_1 = p_2 = \frac{1 - t}{3} \tag{7}$$

and the equilibrium traffic flow is given by

$$x^S_{Duo} = \frac{1 - t}{3(1 + \alpha t)} \tag{8}$$

where the superscript S denotes serial structure.

2.3 Social Optimum

When the total traffic flow is x, the social welfare, defined as the sum of consumer surplus and producer surplus (or firms' profits), is given by

$$S(x) = \int_{1-x}^{1} u\mathrm{d}u - xf(x) = \frac{2x - x^2}{2} - tx - \alpha t x^2 \tag{9}$$

where we assume that commuters with higher value have priority to take the trip. Maximization of the social welfare gives us the socially optimal traffic flow

$$x^S_{Opt} = \frac{1 - t}{1 + 2\alpha t} \tag{10}$$

2.4 Price-free User Equilibrium

Now we consider the price-free scenario, which is commonly studied in the literature. Under such a scenario without any pricing system, the commuters can travel freely on any route subject only to the latency of the route, thus the equilibrium traffic flow will be

$$x_{UE}^{S} = \frac{1-t}{1+\alpha t} \tag{11}$$

which can be easily obtained from equation (4) by letting $p = 0$.

2.5 Monopoly Equilibrium

The last scenario we consider is monopoly. If the two firms merge and become a monopolist by setting the total entrance fee as p, then the profit maximization of the monopolist gives the equilibrium price

$$p = p_1 + p_2 = \frac{1-t}{2} \tag{12}$$

and equilibrium traffic flow

$$x_{Mon}^{S} = \frac{1-t}{2(1+\alpha t)} \tag{13}$$

2.6 Comparison of Traffic Flows

A comparison of the above findings regarding equilibrium traffic flows demonstrates that the following relationship holds.

Lemma 2.1. $x_{Duo}^{S} < x_{Mon}^{S} < x_{Opt}^{S} < x_{UE}^{S} = 2x_{Mon}^{S} = 3x_{Duo}^{S}$.

Lemma 2.1 shows that under the serial structure, traffic flows are smallest in the case of duopoly, followed by the case of monopoly, followed by the social optimum, and the price-free user equilibrium, respectively. We note that intuitively, the socially optimal traffic flow is less than the price-free equilibrium in which commuters can travel without monetary cost on any route that is able to suit their needs, which points to the gap between between social optimum and individual optimization. It is also intuitive that both duopoly and monopoly pricing schemes induce less traffic flow in equilibrium on the single sequential route from A to B to C. Lower traffic flows under duopoly compared to monopoly are driven by the higher total price set by firm 1 and firm 2 collectively through their mutual best response.

2.7 Comparative Statics Analysis

We are also interested in how the equilibrium traffic flows under each scenario will change when the latency parameters change. Based on the results from previous subsections, we can derive the results of comparative statics analysis with respect to parameters t_i and α_i, $i = 1, 2$ and also the composite parameters t and α. The results are summarized in the following proposition and the proof is in the Appendix.

Proposition 2.2. *For every scenario k, $k = Duo, Opt, UE, Mon$, for every segment i, $i = 1, 2$, we have*
(i) $\frac{\partial x_k^S}{\partial t} < 0$;
(ii) $\frac{\partial x_k^S}{\partial \alpha} < 0$;
(iii) $\frac{\partial x_k^S}{\partial t_i} < 0$;
(iv) $\frac{\partial x_k^S}{\partial \alpha_i} < 0$.

Proposition 2.2 states that the traffic flows under each structure are decreasing in the unoccupied travel time for the whole trip t, the unoccupied travel time for each segment t_i, the composite latency parameter for the whole trip α, as well as the latency parameter for each segment α_i, where $i = 1, 2$. Recall that one interpretation for the parameter t (or t_i) is in representing the length of the route. The proposition states that in each scenario we analyzed, equilibrium traffic flows are decreasing in route length, whether for the composite route in the entire travel plan, or the individual route segments. The result with respect to α (or α_i) says that when the latency of any given travel segment increases, including that of the entire travel plan, the equilibrium traffic flow decreases.

3 Benchmark Model II: Parallel Structure

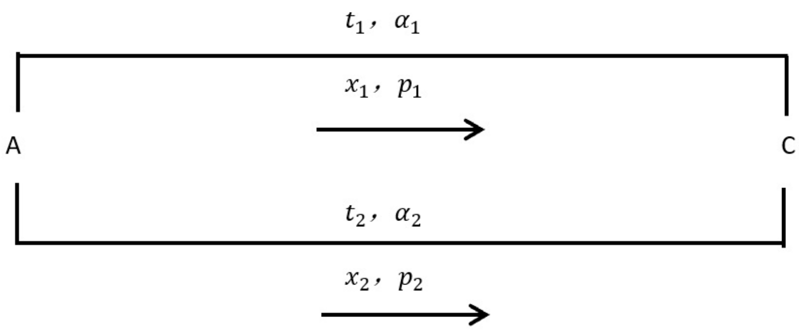
We now analyze the competition between two firms in a transportation route system of parallel structure. Without loss of generality, we focus on interior solutions in equilibrium.

3.1 Settings

Similarly to the setup in Section 2.1, commuters would like to travel from A to C. There are two firms: Firm 1 takes charge of one route from A to C and sets an entrance fee of p_1; Firm 2 takes charge of an alternative route from A to C and sets an entrance fee of p_2. Both firms set their prices strategically and independently to maximize their profits, while facing zero operating costs. All commuters that are willing to travel from A to C must choose exactly one route among the two options and pay the associated entrance fee. A commuter is willing to travel if and only if his/her value of transportation is larger than (or equal to) the entrance fee plus the travel latency. For ease of notation, we again use t_i and α_i for latency parameters for route i ($i = 1, 2$) in the parallel structure, where t_i indicates the trip time on a completely unoccupied route i and α_i is the latency parameter representing the condition of route i.[3] Since commuters' value of transportation

[3]Note that in Section 2, t_i and α_i are latency parameters for segment i in the serial structure. The parameters in the serial structure and those in the parallel structure, though represented by the same symbols, do not necessarily

is no more than 1, we restrict our focus on the reasonable situations in which $max(t_1, t_2) \leq 1$. The settings for the benchmark model of parallel structure can be illustrated by Figure 2.



$$x = x_1 + x_2$$

$$u = max\ \{\theta - p_1 - f(x_1), \theta - p_2 - f(x_2), 0\}$$

$$\pi_1 = x_1 p_1 \qquad\qquad \pi_2 = x_2 p_2$$

Figure 2: Parallel Structure

3.2 Duopoly Equilibrium

Suppose that in equilibrium commuters of size x_1 go through route 1 and commuters of size x_2 go through route 2. Hence, the (marginal) consumer with value $1 - x_1 - x_2$ must be indifferent between the three options: going through route 1, going through route 2, and not traveling, which implies

$$\begin{aligned} 1 - x_1 - x_2 - p_1 - t_1(1 + \alpha_1 x_1) &= 0 \\ 1 - x_1 - x_2 - p_2 - t_2(1 + \alpha_2 x_2) &= 0 \end{aligned}$$

The above equilibrium conditions imply that the two routes for AC have the same total cost:

$$p_1 + t_1(1 + \alpha_1 x_1) = p_2 + t_2(1 + \alpha_2 x_2)$$

and we can express traffic flows as functions of firms' prices:

$$x_1 = \frac{t_2\alpha_2 + p_2 + t_2 - (1 + t_2\alpha_2)(p_1 + t_1)}{t_1\alpha_1 + t_2\alpha_2 + t_1\alpha_1 t_2\alpha_2} \tag{14}$$

$$x_2 = \frac{t_1\alpha_1 + p_1 + t_1 - (1 + t_1\alpha_1)(p_2 + t_2)}{t_1\alpha_1 + t_2\alpha_2 + t_1\alpha_1 t_2\alpha_2} \tag{15}$$

$$x = \frac{1 - \frac{t_1\alpha_1(p_2+t_2)}{t_1\alpha_1+t_2\alpha_2} - \frac{t_2\alpha_2(p_1+t_1)}{t_1\alpha_1+t_2\alpha_2}}{1 + \frac{t_1\alpha_1 t_2\alpha_2}{t_1\alpha_1+t_2\alpha_2}} \tag{16}$$

have the same values.

where $x = x_1 + x_2$ denote the total flow. Hence the profits for firm 1 and firm 2 are given by the following expressions

$$\Pi_1(p_1) = \frac{t_2\alpha_2 + p_2 + t_2 - (1 + t_2\alpha_2)(p_1 + t_1)}{t_1\alpha_1 + t_2\alpha_2 + t_1\alpha_1 t_2\alpha_2} p_1 \tag{17}$$

$$\Pi_2(p_2) = \frac{t_1\alpha_1 + p_1 + t_1 - (1 + t_1\alpha_1)(p_2 + t_2)}{t_1\alpha_1 + t_2\alpha_2 + t_1\alpha_1 t_2\alpha_2} p_2 \tag{18}$$

To maximize firms' profits, the first order conditions are

$$\begin{aligned}
\frac{\partial \Pi_1(p_1)}{\partial p_1} &= \frac{t_2\alpha_2 + p_2 + t_2 - t_1 - t_1 t_2\alpha_2 - 2(1 + t_2\alpha_2)p_1}{t_1\alpha_1 + t_2\alpha_2 + t_1\alpha_1 t_2\alpha_2} = 0 \\
\frac{\partial \Pi_2(p_2)}{\partial p_2} &= \frac{t_1\alpha_1 + p_1 + t_1 - t_2 - t_1 t_2\alpha_1 - 2(1 + t_1\alpha_1)p_2}{t_1\alpha_1 + t_2\alpha_2 + t_1\alpha_1 t_2\alpha_2} = 0
\end{aligned}$$

which jointly determine the equilibrium prices

$$p_1 = \frac{2t_2\alpha_2 + 2t_1\alpha_1 t_2\alpha_2 + t_2 - t_1 + t_1 t_2\alpha_1 - 2t_1^2\alpha_1 - 2t_1 t_2\alpha_2 - 2t_1^2 t_2\alpha_1\alpha_2 + t_1\alpha_1}{4t_1\alpha_1 + 4t_2\alpha_2 + 4t_1\alpha_1 t_2\alpha_2 + 3} \tag{19}$$

$$p_2 = \frac{2t_1\alpha_1 + 2t_1\alpha_1 t_2\alpha_2 + t_1 - t_2 + t_1 t_2\alpha_2 - 2t_2^2\alpha_2 - 2t_1 t_2\alpha_1 - 2t_1 t_2^2\alpha_1\alpha_2 + t_2\alpha_2}{4t_1\alpha_1 + 4t_2\alpha_2 + 4t_1\alpha_1 t_2\alpha_2 + 3} \tag{20}$$

Substituting (19) and (20) into (14)-(16), we obtain the equilibrium traffic flows for each route and for the whole system, respectively.

$$x_1 = \frac{(1 + t_1\alpha_1)(1 + t_2\alpha_2)(2t_2\alpha_2 - 2\alpha_2 t_1 t_2 + t_2 - t_1) + (t_1\alpha_1 + t_1\alpha_1 t_2\alpha_2)(1 - t_1)}{(t_1\alpha_1 + t_2\alpha_2 + t_1\alpha_1 t_2\alpha_2)(4t_1\alpha_1 + 4t_2\alpha_2 + 4t_1\alpha_1 t_2\alpha_2 + 3)} \tag{21}$$

$$x_2 = \frac{(1 + t_1\alpha_1)(1 + t_2\alpha_2)(2t_1\alpha_1 - 2\alpha_1 t_1 t_2 + t_1 - t_2) + (t_2\alpha_2 + t_1\alpha_1 t_2\alpha_2)(1 - t_2)}{(t_1\alpha_1 + t_2\alpha_2 + t_1\alpha_1 t_2\alpha_2)(4t_1\alpha_1 + 4t_2\alpha_2 + 4t_1\alpha_1 t_2\alpha_2 + 3)} \tag{22}$$

$$\begin{aligned}
x_{Duo}^P = & \frac{2(1 + t_1\alpha_1)(1 + t_2\alpha_2)(2t_2\alpha_2 - 2\alpha_2 t_1 t_2 + t_2 - t_1)}{(t_1\alpha_1 + t_2\alpha_2 + t_1\alpha_1 t_2\alpha_2)(4t_1\alpha_1 + 4t_2\alpha_2 + 4t_1\alpha_1 t_2\alpha_2 + 3)} \\
& + \frac{(t_1\alpha_1 + t_1\alpha_1 t_2\alpha_2)(1 - t_1)}{(t_1\alpha_1 + t_2\alpha_2 + t_1\alpha_1 t_2\alpha_2)(4t_1\alpha_1 + 4t_2\alpha_2 + 4t_1\alpha_1 t_2\alpha_2 + 3)} \\
& + \frac{(t_2\alpha_2 + t_1\alpha_1 t_2\alpha_2)(1 - t_2)}{(t_1\alpha_1 + t_2\alpha_2 + t_1\alpha_1 t_2\alpha_2)(4t_1\alpha_1 + 4t_2\alpha_2 + 4t_1\alpha_1 t_2\alpha_2 + 3)}
\end{aligned} \tag{23}$$

where the superscript P denotes parallel.

3.3 Social Optimum

When the total traffic is $x_1 + x_2$, the social welfare is given by

$$\begin{aligned}
S(x_1, x_2) &= \int_{1-x_1-x_2}^{1} u\mathrm{d}u - x_1 f_1(x_1) - x_2 f_2(x_2) \\
&= \frac{1 - (1 - x_1 - x_2)^2}{2} - t_1(1 + \alpha_1 x_1)x_1 - t_2(1 + \alpha_2 x_2)x_2
\end{aligned} \tag{24}$$

with first order conditions

$$
\begin{aligned}
1 - x_1 - x_2 - t_1(1 + 2\alpha_1 x_1) &= 0 \\
1 - x_1 - x_2 - t_2(1 + 2\alpha_2 x_2) &= 0
\end{aligned}
$$

By the above first order conditions, we obtain the socially optimal traffic flows for each route and for the whole system.

$$
\begin{aligned}
x_1 &= \frac{t_2 - t_1 + 2t_2\alpha_2 - 2t_1t_2\alpha_2}{2t_1\alpha_1 + 2t_2\alpha_2 + 4t_1\alpha_1t_2\alpha_2} && (25) \\
x_2 &= \frac{t_1 - t_2 + 2t_1\alpha_1 - 2t_1t_2\alpha_1}{2t_1\alpha_1 + 2t_2\alpha_2 + 4t_1\alpha_1t_2\alpha_2} && (26) \\
x^P_{Opt} &= \frac{t_1\alpha_1 + t_2\alpha_2 - t_1t_2(\alpha_1 + \alpha_2)}{t_1\alpha_1 + t_2\alpha_2 + 2t_1\alpha_1t_2\alpha_2} && (27)
\end{aligned}
$$

3.4 Price-free User Equilibrium

Under the user equilibrium without any pricing system, in which commuters are subject only to the latency of the routes, the equilibrium flows are

$$
\begin{aligned}
x_1 &= \frac{t_2\alpha_2 + t_2 - (1 + t_2\alpha_2)t_1}{t_1\alpha_1 + t_2\alpha_2 + t_1\alpha_1t_2\alpha_2} && (28) \\
x_2 &= \frac{t_1\alpha_1 + t_1 - (1 + t_1\alpha_1)t_2}{t_1\alpha_1 + t_2\alpha_2 + t_1\alpha_1t_2\alpha_2} && (29) \\
x^P_{UE} &= \frac{t_1\alpha_1 + t_2\alpha_2 - t_1t_2(\alpha_1 + \alpha_2)}{t_1\alpha_1 + t_2\alpha_2 + t_1\alpha_1t_2\alpha_2} && (30)
\end{aligned}
$$

which can be easily obtained from equations (14)-(16) by letting $p_1 = p_2 = 0$.

3.5 Monopoly Equilibrium

Suppose that the two firms merge and manage the two routes jointly under a single firm. By total profit maximization, the first order conditions with respect to prices are

$$
\begin{aligned}
\frac{\partial(\Pi_1 + \Pi_2)}{\partial p_1} &= \frac{t_2\alpha_2 + 2p_2 + t_2 - t_1 - t_1t_2\alpha_2 - 2(1 + t_2\alpha_2)p_1}{t_1\alpha_1 + t_2\alpha_2 + t_1\alpha_1t_2\alpha_2} = 0 \\
\frac{\partial(\Pi_1 + \Pi_2)}{\partial p_2} &= \frac{t_1\alpha_1 + 2p_1 + t_1 - t_2 - t_1t_2\alpha_1 - 2(1 + t_1\alpha_1)p_2}{t_1\alpha_1 + t_2\alpha_2 + t_1\alpha_1t_2\alpha_2} = 0
\end{aligned}
$$

which jointly determine the equilibrium prices

$$
\begin{aligned}
p_1 &= \frac{1}{2}(1 - t_1) && (31) \\
p_2 &= \frac{1}{2}(1 - t_2) && (32)
\end{aligned}
$$

Substituting (31) and (32) into (14)-(16), we obtain the monopoly traffic flows for each route and for the whole system, respectively.

$$x_1 = \frac{(\alpha_2+1)t_2 - t_1(\alpha_2 t_2+1)}{2(t_1\alpha_1+t_2\alpha_2+t_1\alpha_1 t_2\alpha_2)} \tag{33}$$

$$x_2 = \frac{(\alpha_1+1)t_1 - t_2(\alpha_1 t_1+1)}{2(t_1\alpha_1+t_2\alpha_2+t_1\alpha_1 t_2\alpha_2)} \tag{34}$$

$$x^P_{Mon} = \frac{t_1\alpha_1+t_2\alpha_2 - t_1 t_2(\alpha_1+\alpha_2)}{2(t_1\alpha_1+t_2\alpha_2+t_1\alpha_1 t_2\alpha_2)} \tag{35}$$

3.6 Comparison of Traffic Flows

The following relationship holds with regard to the traffic flows in the parallel structure across 4 different scenarios considered:

Lemma 3.1. $x^P_{Mon} < x^P_{Duo} < x^P_{Opt} < x^P_{UE} = 2x^P_{Mon}$.

While the total traffic flow ordering between the case of price-free scenario and the social planner's solution remains the same compared to the case of the serial structure, the duopoly leads to a higher traffic flow than monopoly in the parallel structure setting. In the case of parallel route competition between duopolists, two routes are available to commuters under prices which directly compete with one another, which raises the joint traffic flow beyond that of the monopolist.

3.7 Comparative Statics Analysis

In this subsection, we analyze how the equilibrium traffic flows under each scenario change in the latency parameters t_i and α_i, $i = 1, 2$. The results are summarized in the following proposition.

Proposition 3.2. *For every scenario k, $k = Duo, Opt, UE, Mon$, for every route i, $i = 1, 2$ with $-i = 2 - i$, we have*
(i) $\frac{\partial x^P_k}{\partial t_i} < 0$;
(ii) $\frac{\partial x^P_k}{\partial \alpha_i} < 0$.

The proposition states that traffic flows in the parallel structure for any of the market structure scenarios are decreasing in the latency parameters t_i and α_i of each firm's segment. The proof is provided in the Appendix. The reasoning is that an increase in the latency parameter(s) on any firm's segment results in an increased realized latency, or in other words slower travel time on their segment. This serves to reduce the number of commuters who find traveling on that segment appealing. While some of the commuters will switch to traveling on the other firm's route, the increase in the latency of the other firm's route due to their increased number of commuters traveling on it will deter some of the original commuters from traveling at all. The result is a lower total traffic flow in the system.

4 Comparison Between Serial and Parallel Structures

We now compare the cases of serial and parallel structures described previously. For ease of comparison, we restrict our attention to the case that firm 1 and firm 2 under a given structure are identical in their price competition (by setting $\alpha_1^j = \alpha_2^j$ and $t_1^j = t_2^j$ for $j = S, P$) (the symmetry condition), which simplifies the expressions substantially. To make the comparison reasonable, we also impose another equivalence condition that the two structures under the price-free scenario should have the same equilibrium flow.[4]

4.1 Analogy between the Circuit System and the Transportation System

We utilize some established results from physics to help us compare the equilibrium outcomes under the serial and parallel structures. In this subsection, we provide two figures to describe the analogy between a circuit system and a transportation system. The detailed explanations are offered in the Appendix.

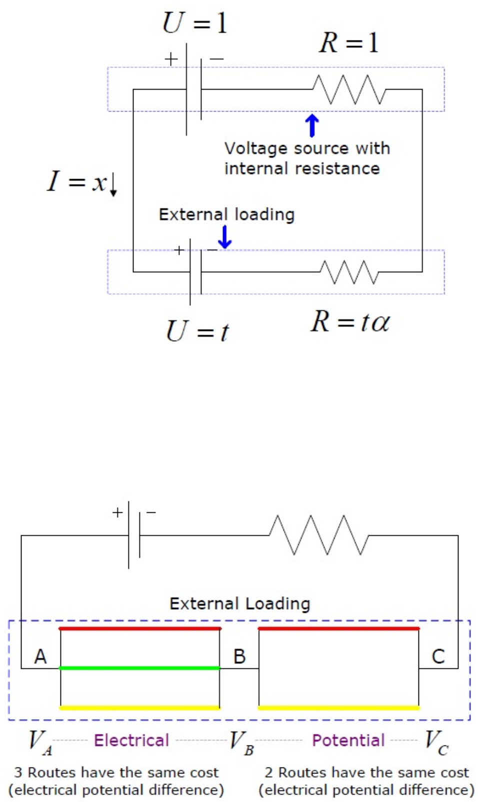


Figure 3: Analogy between the Circuit System and the Transportation System

[4]The equivalence condition requires $t_1^S + t_2^S = \frac{t_1^P t_2^P (\alpha_1^P + \alpha_2^P)}{t_1^P \alpha_1^P + t_2^P \alpha_2^P}$ and $t_1^S \alpha_1^S + t_2^S \alpha_2^S = \frac{t_1^P \alpha_1^P t_2^P \alpha_2^P}{t_1^P \alpha_1^P + t_2^P \alpha_2^P}$, which can be obtained by comparing the two graphs in right panel in Figure 4.

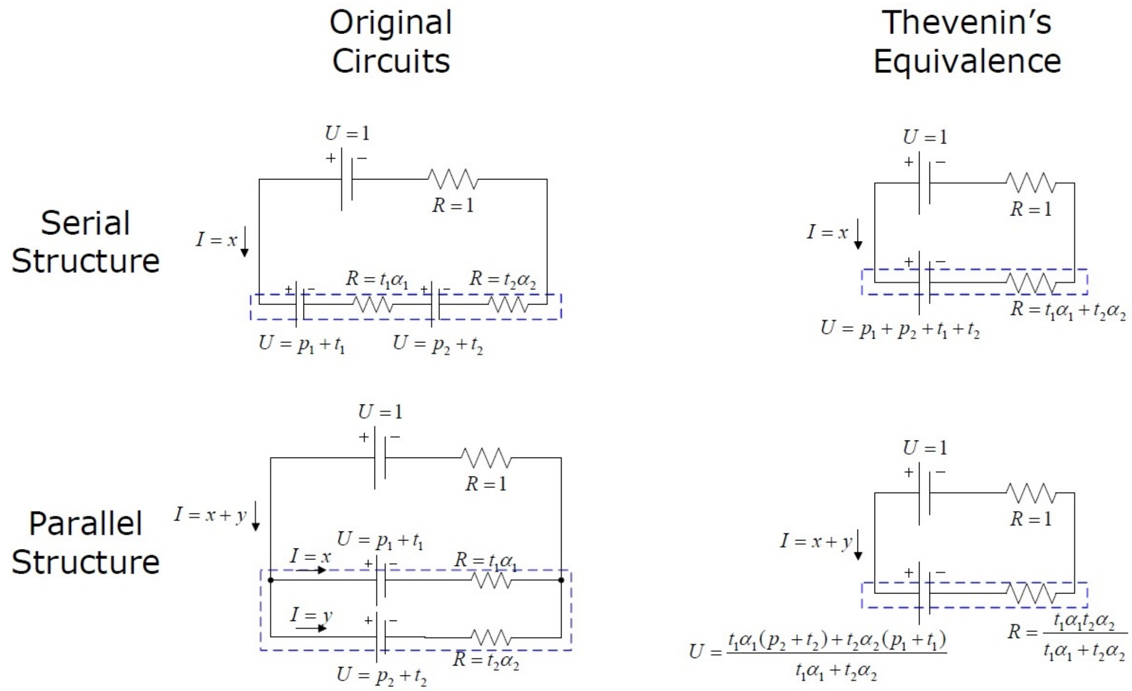


Figure 4: Thévenin's Theorem

We can interpret the expression of flow x (see equation (4) for example) as follows. Assume αt is the resistance for the route with latency $f(x) = t(1+\alpha x)$, then the conductance is the reciprocal of the resistance. Thus, the equivalent conductance of the parallel structure takes the summation of the two routes, while the equivalent resistance of the serial structure takes the summation of the two segments. Hence, the denominator in the expression of x can be expressed as 1 plus total resistance. We define the idle cost for the route with latency $f(x) = t(1 + \alpha x)$ and entrance fee p as $p+t$. Idle cost takes the conductance-weighted average when connected in the parallel structure and takes the summation when connected in the serial structure. Therefore, the numerator in the expression of x for the parallel structure should equal 1 minus total conductance-weighted idle cost. Note that the following relationship is satisfied in equilibrium:

$$\text{Flow} = 1 - \text{IdleCost} - \text{FlowCost}$$

where FlowCost = Flow × Resistance. Equivalently, we have

$$\text{Flow} = \frac{1 - \text{IdleCost}}{1 + \text{Resistance}} \tag{36}$$

4.2 Equilibrium Flow

We can use Thévenin's theorem to characterize the equivalence for the transportation system. Assume the latency function for the serial structure is $f(x) = t(1+\alpha x)$, then the associated latency function for both routes in the parallel structure is $f_1(x) = f_2(x) = t(1 + 2\alpha x)$. Given that the equilibrium outcomes are the same in the price-free scenario under both structures, it is easy to verify that both structures are also outcome equivalent (including demand and social welfare) in the social optimum and monopoly scenarios. The equilibrium traffic flows under both structures across different scenarios are summarized in the following table.

Table 1: Traffic Flows

	Serial	Parallel
User Equilibrium	$\frac{1-t}{1+\alpha t}$	$\frac{1-t}{1+\alpha t}$
Social Optimum	$\frac{1-t}{1+2\alpha t}$	$\frac{1-t}{1+2\alpha t}$
Monopoly	$\frac{1-t}{2(1+\alpha t)}$	$\frac{1-t}{2(1+\alpha t)}$
Duopoly	$\frac{1-t}{3(1+\alpha t)}$	$\frac{(1-t)(1+2\alpha t)}{(1+\alpha t)(4\alpha t+1)}$

Using x^S_{Duo} to denote traffic flows under the duopoly context in the serial structure, x^P_{Duo} to denote that under the parallel structure, and $x^{S/P}_{Mon}, x^{S/P}_{Opt}, x^{S/P}_{UE}$ to denote the identical traffic flow under monopoly, social optimum and user equiibrium respectively, by Lemmas 2.1 and 3.1, we have the following proposition.

Proposition 4.1. *Under symmetry and equivalence conditions, the equilibrium flows under different scenarios and different structures have the following relationship:*
$x^S_{Duo} < x^{S/P}_{Mon} < x^P_{Duo} < x^{S/P}_{Opt} < x^{S/P}_{UE}$.

Proposition 4.1 implies that the symmetric duopoly competition under the parallel structure results in a larger traffic flow than that under the serial structure if the monopoly traffic flows under these two structures are the same.

4.3 Equilibrium Price and Profit

We now consider the price and profit expressions for the scenarios of monopoly and duopoly under the serial and parallel structures. By the equivalence condition, the case of monopoly is identical across the serial and parallel structures in terms of both price and profit. However, the duopoly case results in different price and profit levels under these two structures. The results are summarized in Tables 2 and 3.

Table 2: Price

	Serial	Parallel
Monopoly	$p_1 + p_2 = \frac{1-t}{2}$	$p_1 = p_2 = \frac{1-t}{2}$
Duopoly	$p_1 = p_2 = \frac{1-t}{3}$	$p_1 = p_2 = \frac{2\alpha t(1-t)}{1+4\alpha t}$

Table 3: Profit

	Serial	Parallel
Monopoly	$\Pi = \frac{(1-t)^2}{4(1+\alpha t)}$	$\Pi = \frac{(1-t)^2}{4(1+\alpha t)}$
Duopoly	$\Pi_1 = \Pi_2 = \frac{(1-t)^2}{9(1+\alpha t)}$	$\Pi_1 = \Pi_2 = \frac{\alpha t(1+2\alpha t)(1-t)^2}{(1+4\alpha t)^2(1+\alpha t)}$

In terms of comparing the profit levels between the parallel and serial structures, the result depends on the values of parameters α and t and $\alpha t = \frac{1}{2}$ serves as the cutoff point. A full characterization of duopoly profit comparison is stated in the following proposition.

Proposition 4.2. *Under symmetry and equivalence conditions, for the duopoly scenario, the relationship between profit under serial and parallel structures depends on the parameter values. If $\alpha t > \frac{1}{2}$, the profit under the parallel structure is higher; If $\alpha t < \frac{1}{2}$, the profit under the serial structure is higher; If $\alpha t = \frac{1}{2}$, the profits under both structures are the same.*

Proposition 4.2 implies that when the route condition is poor (such that α is large) or the travel time on the unoccupied road is long (such that t is large), the profit is higher under the parallel structure. However, when the route condition is sufficiently good or the ideal travel time is sufficiently short, the serial structure can generate a higher profit.

4.4 Equilibrium Surplus

The consumer surplus and social surplus can also be calculated under both serial and parallel structures in the symmetric case. When total traffic flow is x, both structures have the same social surplus functional form $\frac{2x-x^2}{2} - tx - t\alpha x^2$, and consumer surplus is simply the difference between the social surplus and firms' profits. The results for consumer surplus and social surplus under both structures between the monopoly scenario and duopoly scenario are shown in Table 4 and Table 5, respectively.

Table 4: Consumer Surplus

	Serial	Parallel
Monopoly	$\frac{(1-t)^2}{8(1+\alpha t)^2}$	$\frac{(1-t)^2}{8(1+\alpha t)^2}$
Duopoly	$\frac{(1-t)^2}{18(1+\alpha t)^2}$	$\frac{(1-t)^2(1+2\alpha t)^2}{2(1+\alpha t)^2(1+4\alpha t)^2}$

Table 5: Social Surplus

	Serial	Parallel
Monopoly	$\frac{(2\alpha t+3)(1-t)^2}{8(1+\alpha t)^2}$	$\frac{(2\alpha t+3)(1-t)^2}{8(1+\alpha t)^2}$
Duopoly	$\frac{(4\alpha t+5)(1-t)^2}{18(1+\alpha t)^2}$	$\frac{(1-t)^2(1+2\alpha t)(4\alpha^2 t^2+6\alpha t+1)}{2(1+\alpha t)^2(1+4\alpha t)^2}$

Note that in Table 5, the duopoly scenario generates a lower social surplus than the monopoly scenario under the serial structure, and the reverse relation holds for the parallel structure. Such different efficiency results under these two structures are driven by the different equilibrium traffic flow levels between the monopoly and duopoly scenarios (See Proposition 4.1 for details). Since the social surplus function is increasing in traffic flow up to the socially optimal flow level, a lower flow in duopoly than in monopoly under the serial structure leads to a lower efficiency level in duopoly compared to the monopoly scenario. For the parallel structure, the reverse result is due to higher flow in duopoly than in monopoly.

Based on the results in Tables 4 and 5, a more important question we would like to ask is whether the serial structure or the parallel structure brings higher surplus to the commuters and to society. We address this question in the following proposition.

Proposition 4.3. *Under symmetry and equivalence conditions, for the duopoly scenario, both consumer surplus and social surplus are higher under the parallel structure than under the serial structure.*

The proposition implies that regardless of the route condition and the travel time, from commuters' perspective as well as from a social welfare perspective, the parallel structure is strictly preferred to the serial structure.

5 Extension I: Operating Cost

Up to this point, we have assumed that firms' operating costs are negligible, and thus that firms are essentially revenue maximizing. In this extension, we relax this assumption, and allow the firms to have operating costs that could differ based upon some exogenous route characteristics. For example, operating costs may be a function of route length and/or route condition which are both exogenous parameters in our model, $C_1(t_1, \alpha_1), C_2(t_2, \alpha_2)$. Such a framework applies well to situations in which maintenance is primarily a function of distance traveled, which can often be the case for subway systems, buses, high speed rail, and airplane transport.

5.1 Serial Structure

We first examine the case of the serial structure. Given entrance fees p_1, p_2, the traffic flow is given by $x = \frac{1-p_1-p_2-t_1-t_2}{1+t_1\alpha_1+t_2\alpha_2}$.

Hence, the profits for each firm are given by

$$\begin{aligned}
\Pi_1(p_1) &= \frac{1-p_1-p_2-t_1-t_2}{1+t_1\alpha_1+t_2\alpha_2}(p_1 - C_1) \\
\Pi_2(p_2) &= \frac{1-p_1-p_2-t_1-t_2}{1+t_1\alpha_1+t_2\alpha_2}(p_2 - C_2)
\end{aligned}$$

with first order conditions

$$\begin{aligned}
\frac{\partial \Pi_1(p_1)}{\partial p_1} &= \frac{1 - 2p_1 - p_2 - t_1 - t_2 + C_1}{1 + t_1\alpha_1 + t_2\alpha_2} = 0 \\
\frac{\partial \Pi_2(p_2)}{\partial p_2} &= \frac{1 - p_1 - 2p_2 - t_1 - t_2 + C_2}{1 + t_1\alpha_1 + t_2\alpha_2} = 0
\end{aligned}$$

yielding equilibrium prices

$$\begin{aligned}
p_1 &= p_1^* + \frac{2C_1 - C_2}{3} && (37) \\
p_2 &= p_2^* + \frac{2C_2 - C_1}{3} && (38)
\end{aligned}$$

where $p_1^* = p_2^* = \frac{1-t_1-t_2}{3}$ denotes the previously derived equilibrium prices without any operating costs (equation (7)). Hence, depending on the relative magnitudes of costs among the two firms, the equilibrium price with operating costs may be greater than or less than the equilibrium price without operating costs. Note that at least one firm's price under operating costs will exceed its price without operating costs.

The equilibrium traffic flow is

$$x = \frac{1 - t_1 - t_2 - (C_1 + C_2)}{3(1 + t_1\alpha_1 + t_2\alpha_2)} \tag{39}$$

which is indeed lower than the equilibrium traffic flow without operating costs.

5.2 Parallel Structure

For the case of the parallel structure, the profits for each firm are given by

$$\begin{aligned}
\Pi_1(p_1) &= \frac{t_2\alpha_2 + p_2 + t_2 - (1 + t_2\alpha_2)(p_1 + t_1)}{t_1\alpha_1 + t_2\alpha_2 + t_1\alpha_1 t_2\alpha_2}(p_1 - C_1) \\
\Pi_2(p_2) &= \frac{t_1\alpha_1 + p_1 + t_1 - (1 + t_1\alpha_1)(p_2 + t_2)}{t_1\alpha_1 + t_2\alpha_2 + t_1\alpha_1 t_2\alpha_2}(p_2 - C_2)
\end{aligned}$$

The first order conditions (ignoring the common denominators) give us

$$\begin{aligned}
\frac{\partial \Pi_1(p_1)}{\partial p_1} &= t_2\alpha_2 + p_2 + t_2 - t_1 - t_1 t_2\alpha_2 - 2(1 + t_2\alpha_2)p_1 + (1 + t_2\alpha_2)C_1 = 0 \\
\frac{\partial \Pi_2(p_2)}{\partial p_2} &= t_1\alpha_1 + p_1 + t_1 - t_2 - t_1 t_2\alpha_1 - 2(1 + t_1\alpha_1)p_2 + (1 + t_1\alpha_1)C_2 = 0
\end{aligned}$$

with the pricing solution

$$\begin{aligned}
p_1 &= p_1^* + \frac{(1 + t_2\alpha_2)(C_1 + 2C_2 + 2t_1\alpha_1 C_2)}{4t_1\alpha_1 + 4t_2\alpha_2 + 4t_1\alpha_1 t_2\alpha_2 + 3} && (40) \\
p_2 &= p_2^* + \frac{(1 + t_1\alpha_1)(2C_1 + C_2 + 2t_2\alpha_2 C_1)}{4t_1\alpha_1 + 4t_2\alpha_2 + 4t_1\alpha_1 t_2\alpha_2 + 3} && (41)
\end{aligned}$$

where p_1^*, p_2^* denote equilibrium prices without any operating costs as derived previously (equations (19) and (20)). Therefore, equilibrium prices are strictly higher when introducing operating costs. By contrast, equilibrium prices in the serial structure may be higher or lower than the equilibrium prices without operating costs, depending on the relative magnitudes of firms' operating costs.

Since total traffic flow is expressed as

$$\frac{1 - \frac{t_1\alpha_1(p_2+t_2)}{t_1\alpha_1+t_2\alpha_2} - \frac{t_2\alpha_2(p_1+t_1)}{t_1\alpha_1+t_2\alpha_2}}{1 + \frac{t_1\alpha_1 t_2\alpha_2}{t_1\alpha_1+t_2\alpha_2}}$$

the higher prices under operating costs lead to a lower equilibrium traffic flow.

5.3 Comparison with No Operating Costs

Comparing the analysis with operation costs to the baseline case without operational costs, we observe that operational costs reduce the equilibrium traffic flow for both the parallel and serial structures. In terms of pricing, the result depends on the transportation structure. In the case of the parallel structure, prices are higher for both firms due to the operating cost. For the case of the serial structure, the price for any given firm may be either higher or lower than the situation without operating costs, where the result depends on the relative costs between the two firms. However, at least one of the firms (and potentially both) implements a higher price than in the setup without operating costs, thus reducing the equilibrium traffic flow.

5.4 Comparison Between Serial and Parallel Structures

We now compare the cases of serial and parallel structures with operating costs. As previously, we restrict our attention to the symmetric case of firm 1 and firm 2 in their price competition and impose the equivalence condition, both of which simplify the expressions substantially. In addition, we assume that each firm incurs the same operating cost, $C_1 = C_2 = C$.

5.4.1 Equilibrium Flow

Under the symmetry and equivalence conditions, the latency function for the serial structure is $f(x) = t(1 + \alpha x)$, while the associated latency function for both roads in the parallel structure is $f(x) = t(1 + 2\alpha x)$. Both structures are outcome equivalent in the price-free equilibrium as well as in the social optimum scenario.

When considering the scenario of monopoly, we need to set the total operating cost of the system equal for each case for comparison purposes. Assume the unit operating cost for the monopoly is C, then when analyzing the duopoly context, the corresponding costs should be $C_1 = C_2 = \frac{C}{2}$ for each road in the serial structure, and $C_1 = C_2 = C$ for each road in the parallel structure. With such symmetry and equivalence conditions on operating costs, the scenario of monopoly yields identical traffic flows under the contexts of serial and parallel structures.

For the case of duopoly, the traffic flow is higher under the parallel structure than the serial structure. The equilibrium traffic flows are summarized in the following table and their relations are stated in Proposition 5.1.

Table 6: Traffic Flow

	Serial	Parallel
User Equilibrium	$\frac{1-t}{1+\alpha t}$	$\frac{1-t}{1+\alpha t}$
Social Optimum	$\frac{1-t}{1+2\alpha t}$	$\frac{1-t}{1+2\alpha t}$
Monopoly	$\frac{1-t-C}{2(1+\alpha t)}$	$\frac{1-t-C}{2(1+\alpha t)}$
Duopoly	$\frac{1-t-C}{3(1+\alpha t)}$	$\frac{(1-t-C)(1+2\alpha t)}{(1+\alpha t)(4\alpha t+1)}$

Proposition 5.1. *Under symmetry and equivalence conditions with operating costs, the equilibrium flows under different scenarios and different structures have the following relationship:* $x^{S}_{Duo} < x^{S/P}_{Mon} < x^{P}_{Duo} < x^{S/P}_{Opt} < x^{S/P}_{UE}$.

5.4.2 Equilibrium Price and Profit

In terms of pricing, the monopolist's price for each route in the case of the parallel structure is equivalent to the monopolist's total price over the two segments in the serial structure. For the case of duopoly, the parallel structure has a higher equilibrium price for a sufficiently high operational cost, the cutoff being a function of α and t. Note that our previous condition $\alpha t > \frac{1}{2}$ in the case of no operational cost is sufficient to satisfy this condition for any positive C.

Table 7: Price

	Serial	Parallel
Monopoly	$p_1 + p_2 = \frac{1-t}{2} + \frac{C}{2}$	$p_1 = p_2 = \frac{1-t}{2} + \frac{C}{2}$
Duopoly	$p_1 = p_2 = \frac{1-t}{3} + \frac{C}{6}$	$p_1 = p_2 = \frac{2\alpha t(1-t)}{1+4\alpha t} + \frac{(1+2\alpha t)C}{1+4\alpha t}$

Table 8: Profit

	Serial	Parallel
Monopoly	$\Pi = \frac{(1-t-C)^2}{4(1+\alpha t)}$	$\Pi = \frac{(1-t-C)^2}{4(1+\alpha t)}$
Duopoly	$\Pi_1 = \Pi_2 = \frac{(1-t-C)^2}{9(1+\alpha t)}$	$\Pi_1 = \Pi_2 = \frac{\alpha t(1+2\alpha t)(1-t-C)^2}{(1+4\alpha t)^2(1+\alpha t)}$

The monopolist's profit is the same whether in the serial context or the parallel context. Duopoly profit equals half of equilibrium flow multiplied by the duopoly price. When including operating costs, the $(1-t)^2$ terms in the numerator of the profit expression without operating cost, change to $(1-t-C)^2$. Therefore, the profit comparison between structures is again defined by the cutoff $\alpha t = \frac{1}{2}$, and is fully characterized in Proposition 5.2.

Proposition 5.2. *Under symmetry and equivalence conditions with operating costs, for the duopoly scenario, the relationship between profits under serial and parallel structures depends on the parameter values. If $\alpha t > \frac{1}{2}$, the profit under the parallel structure is higher; If $\alpha t < \frac{1}{2}$, the profit under the serial structure is higher; If $\alpha t = \frac{1}{2}$, the profits under both structures are the same.*

5.4.3 Equilibrium Surplus

Similarly, when including operating costs, the $(1-t)^2$ terms in the numerator of the surplus expressions without operating cost, change to $(1-t-C)^2$, as shown in Tables 9 and 10. Thus, both consumer surplus and social surplus are higher under the parallel structure, and such a result is described in Proposition 5.3.

Table 9: Consumer Surplus

	Serial	Parallel
Monopoly	$\frac{(1-t-C)^2}{8(1+\alpha t)^2}$	$\frac{(1-t-C)^2}{8(1+\alpha t)^2}$
Duopoly	$\frac{(1-t-C)^2}{18(1+\alpha t)^2}$	$\frac{(1-t-C)^2(1+2\alpha t)^2}{2(1+\alpha t)^2(1+4\alpha t)^2}$

Table 10: Social Surplus

	Serial	Parallel
Monopoly	$\frac{(2\alpha t+3)(1-t-C)^2}{8(1+\alpha t)^2}$	$\frac{(2\alpha t+3)(1-t-C)^2}{8(1+\alpha t)^2}$
Duopoly	$\frac{(4\alpha t+5)(1-t-C)^2}{18(1+\alpha t)^2}$	$\frac{(1-t-C)^2(1+2\alpha t)(4\alpha^2 t^2+6\alpha t+1)}{2(1+\alpha t)^2(1+4\alpha t)^2}$

Proposition 5.3. *Under symmetry and equivalence conditions with operating costs, for the duopoly scenario, both consumer surplus and social surplus are higher under the parallel structure than under the serial structure.*

6 Extension II: More than Two Firms

We now allow for the possibility that there are n competitors in charge of n route segments under the cases of serial and parallel structures. For ease of comparison, we restrict our attention to the case that the n firms are symmetric in their price competition, which simplifies the expressions substantially. The analysis for the serial structure is a straightforward generalization of that in Section 2 by defining $t = \sum_{k=1}^{n} t_i$ and $\alpha = \frac{\sum_{k=1}^{n} t_i \alpha_i}{\sum_{k=1}^{n} t_i}$, and is hence skipped. The analysis for the parallel structure is included in the Appendix.

6.1 Equilibrium Flow

Assume the latency function for the serial structure is $f(x) = t(1 + \alpha x)$, then the associated latency function for all n routes in the parallel structure is $f_i(x) = t(1+n\alpha x)$. Similar to the previous section, the parallel and serial structures are outcome equivalent in the price-free equilibrium, social optimum and monopoly. The following lemma solves for the equilibrium.

Lemma 6.1. *For n routes in the parallel structure with latency function $f_i(x) = t(1 + n\alpha x)$, (1) Equilibrium traffic flow for each route is $\frac{(1-t)(n\alpha t+n-1)}{n(1+\alpha t)(2n\alpha t+n-1)}$. (2) Equilibrium prices are identical and equal $\frac{n\alpha t}{2n\alpha t+n-1}$.*

If all other routes set the equilibrium price, then it is optimal for one route to set the equilibrium price as well. The equilibrium traffic flows are summarized in the following table.

Table 11: Traffic Flows

	Serial	Parallel
User Equilibrium	$\frac{1-t}{1+\alpha t}$	$\frac{1-t}{1+\alpha t}$
Social Optimum	$\frac{1-t}{1+2\alpha t}$	$\frac{1-t}{1+2\alpha t}$
Monopoly	$\frac{1-t}{2(1+\alpha t)}$	$\frac{1-t}{2(1+\alpha t)}$
Oligopoly	$\frac{1-t}{(n+1)(1+\alpha t)}$	$\frac{(1-t)(n\alpha t+n-1)}{(1+\alpha t)(2n\alpha t+n-1)}$

Using the same notations for equilibrium flows as in Section 4.2, we have the following proposition.

Proposition 6.2. *Under symmetry and equivalence conditions, the equilibrium traffic flows under different scenarios and different structures have the following relationship:*
$x^S_{Oli} < x^{S/P}_{Mon} < x^P_{Oli} < x^{S/P}_{Opt} < x^{S/P}_{UE}$.

As the number of firms n grows, the traffic flow of the oligopoly market increases under the parallel structure while it decreases under the serial structure. When the number of firms approaches infinity, the traffic flow in the oligopoly market converges to the socially optimal outcome under the parallel structure while it converges to zero under the serial structure.

6.2 Equilibrium Price and Profit

Similarly, for equilibrium prices and profits, our main results from the case of two firms in Section 4.3 can be generalized to the n-firm case. The detailed analysis is included in the Appendix. The expressions for price and profit under monopoly and oligopoly in the serial and parallel structures are provided in Tables 12 and 13.

Proposition 6.3. *Under symmetry and equivalence conditions, for the oligopoly scenario, the relationship between Oligopoly profits under serial and parallel structures depends on the parameter*

Table 12: Price

	Serial	Parallel
Monopoly	$\sum p = \frac{1-t}{2}$	$p_i = \frac{1-t}{2}$
Oligopoly	$p_i = \frac{1-t}{n+1}$	$p_i = \frac{n\alpha t(1-t)}{2n\alpha t+n-1}$

Table 13: Profit

	Serial	Parallel
Monopoly	$\Pi = \frac{(1-t)^2}{4(1+\alpha t)}$	$\Pi = \frac{(1-t)^2}{4(1+\alpha t)}$
Oligopoly	$\Pi_i = \frac{(1-t)^2}{(n+1)^2(1+\alpha t)}$	$\Pi_i = \frac{\alpha t(n\alpha t+n-1)(1-t)^2}{(2n\alpha t+n-1)^2(1+\alpha t)}$

values. If $\alpha t > \frac{1}{n}$, the profit under the parallel structure is higher; If $\alpha t < \frac{1}{n}$, the profit under the serial structure is higher; If $\alpha t = \frac{1}{n}$, the profit under both structures are the same.

The above proposition implies that when the route condition is poor (such that α is large) or the ideal travel time is long (such that t is large), the oligopolist profit is higher under the parallel structure. Additionally, the threshold is decreasing with n, implying that the oligopolist profit is more likely to be higher under the parallel structure as the number of competitors grows.

6.3 Equilibrium Surplus

The consumer surplus and social surplus can also be calculated under both serial and parallel structures in the symmetric case, shown in Tables 14 and 15, respectively.

Table 14: Consumer Surplus

	Serial	Parallel
Monopoly	$\frac{(1-t)^2}{8(1+\alpha t)^2}$	$\frac{(1-t)^2}{8(1+\alpha t)^2}$
Oligopoly	$\frac{(1-t)^2}{2(n+1)^2(1+\alpha t)^2}$	$\frac{(1-t)^2(n\alpha t+n-1)^2}{2(1+\alpha t)^2(2n\alpha t+n-1)^2}$

Table 15: Social Surplus

	Serial	Parallel
Monopoly	$\frac{(2\alpha t+3)(1-t)^2}{8(1+\alpha t)^2}$	$\frac{(2\alpha t+3)(1-t)^2}{8(1+\alpha t)^2}$
Oligopoly	$\frac{(2n\alpha t+2n+1)(1-t)^2}{2(n+1)^2(1+\alpha t)^2}$	$\frac{(1-t)^2(n\alpha t+n-1)(2n\alpha^2t^2+3n\alpha t+n-1)}{2(1+\alpha t)^2(2n\alpha t+n-1)^2}$

Our analysis shows that the result that the parallel structure dominates the serial structure by welfare criteria in the case of duopoly still holds when there are more than two firms (See the Appendix for detailed analysis). We summarize this result in the following proposition.

Proposition 6.4. *Under symmetry and equivalence conditions, for the oligopoly scenario, both consumer surplus and social surplus are higher under parallel structure than under serial structure.*

The above proposition indicates that the welfare comparison result that the parallel structure is strictly preferred to the serial structure from a consumer and social welfare perspective, is robust for the case of oligopoly.

7 Extension III: Hybrid Market Structures

In our benchmark model, we have considered situations where a single market structure (duopoly, monopoly) applies to the entire transportation structure. In other words, in the benchmark analysis we have considered a serial duopoly, serial monopoly, parallel duopoly, and parallel monopoly. In this extension we consider the possibility that market structure may vary between the travel segments A to B and B to C. We analyze two hybrid structures.

In the first structure, which is a serial duopoly-monopoly hybrid, there is duopoly competition over the segment A to B, while the segment B to C is solely operated by one of the two firms. Since commuters decide whether to take the trip or not based on their valuation of the complete travel path A to C, this analysis is identical to the reverse case in which A to B is operated by a monopolist, while B to C has duopoly competition.

In the second structure, which is a serial-parallel duopoly hybrid, two firms engage in duopoly competition over the segment A to B, as well as over the segment B to C.

7.1 Duopoly in AB and Monopoly in BC

We first consider the market structure where two firms compete for route AB by setting an entrance fee of p_1, p_2 respectively, and firm 1 also takes charge of the route BC by setting an entrance fee of q. The setting for this hybrid structure can be illustrated by Figure 5.

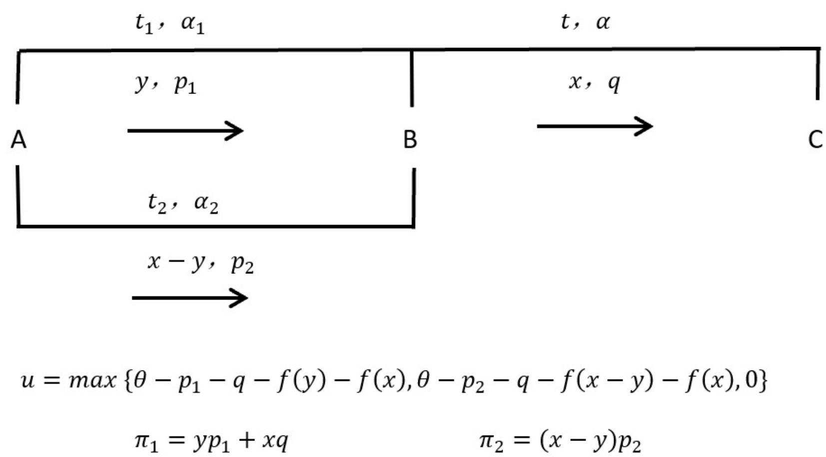


Figure 5: Duopoly in AB and Monopoly in BC

In equilibrium, suppose the total traffic flow is denoted by x while y of them choose firm 1 over the A to B segment. Thus, the (marginal) consumer with valuation $1-x$ must be indifferent between three options: (1) not traveling, (2) traveling through firm 1 over AB, and (3) traveling through firm 2 over AB. This equilibrium condition implies the following two equations.

$$\begin{aligned} 1 - x - p_1 - q - t_1(1+\alpha_1 y) - t(1+\alpha x) &= 0 \\ 1 - x - p_2 - q - t_2(1+\alpha_2(x-y)) - t(1+\alpha x) &= 0 \end{aligned}$$

Expressing the flow variables x and y in terms of price variables p_1, p_2, and q, we obtain

$$x = \frac{1-(q+t) - \frac{t_1\alpha_1}{t_1\alpha_1+t_2\alpha_2}(p_2+t_2) - \frac{t_2\alpha_2}{t_1\alpha_1+t_2\alpha_2}(p_1+t_1)}{1+t\alpha+\frac{t_1\alpha_1 t_2\alpha_2}{t_1\alpha_1+t_2\alpha_2}} \tag{42}$$

$$y = \frac{t_2\alpha_2}{t_1\alpha_2+t_2\alpha_2}x + \frac{p_2+t_2-p_1-t_1}{t_1\alpha_1+t_2\alpha_2} \tag{43}$$

$$x - y = \frac{t_1\alpha_1}{t_1\alpha_2+t_2\alpha_2}x + \frac{p_1+t_1-p_2-t_2}{t_1\alpha_1+t_2\alpha_2} \tag{44}$$

Note that the profit functions for two firms are given by

$$\Pi_1 = yp_1 + xq \tag{45}$$

$$\Pi_2 = (x-y)p_2 \tag{46}$$

Substituting (42)-(44) into (45) and (46), we have the following first order conditions by profit maximization.

$$\begin{aligned} \frac{\partial \Pi_1}{\partial p_1} = 0 \quad &\Rightarrow \quad (-2p_1+p_2-t_1+t_2)(\alpha t+1) - \alpha_2 t_2(2p_1+2q+t+t_1-1) = 0 \\ \frac{\partial \Pi_1}{\partial q} = 0 \quad &\Rightarrow \quad \alpha_1(-t_1)(p_2+2q+t+t_2-1) - \alpha_2 t_2(2p_1+2q+t+t_1-1) = 0 \\ \frac{\partial \Pi_2}{\partial p_2} = 0 \quad &\Rightarrow \quad (p_1-2p_2+t_1-t_2)(\alpha t+1) - \alpha_1 t_1(2p_2+q+t+t_2-1) = 0 \end{aligned}$$

Equilibrium prices are given by the following expressions

$$p_1 = -\frac{2(t_1 - t_2)(\alpha t + 1) + \alpha_1 t_1(t + 3t_1 - 2t_2 - 1)}{6(\alpha t + \alpha_1 t_1 + 1)} \tag{47}$$

$$p_2 = \frac{(t_1 - t_2)(\alpha t + 1) - \alpha_1 t_1(t + t_2 - 1)}{3(\alpha t + \alpha_1 t_1 + 1)} \tag{48}$$

$$q = -\frac{(3t + t_1 + 2t_2 - 3)(\alpha t + 1) + 2\alpha_1 t_1(t + t_2 - 1)}{6(\alpha t + \alpha_1 t_1 + 1)} \tag{49}$$

Substituting (47)-(49) into (42) and (43), we obtain the equilibrium traffic flows, where $x^* - y^*$ denotes the traffic flow on firm 2's route over the AB segment.

$$x^* = \frac{\alpha_1 t_1((3t + t_1 + 2t_2 - 3)(-(\alpha t + 1)) - 2\alpha_1 t_1(t + t_2 - 1)) - 3\alpha_2(t + t_1 - 1)t_2(\alpha t + \alpha_1 t_1 + 1)}{6(\alpha t + \alpha_1 t_1 + 1)(\alpha_2 t_2(\alpha t + 1) + \alpha_1 t_1(\alpha t + \alpha_2 t_2 + 1))} \tag{50}$$

$$y^* = \frac{(\alpha t + 1)(-2(t_1 - t_2)(\alpha t + 1) - \alpha_1 t_1(t + 3t_1 - 2t_2 - 1)) - 3\alpha_2(t + t_1 - 1)t_2(\alpha t + \alpha_1 t_1 + 1)}{6(\alpha t + \alpha_1 t_1 + 1)(\alpha_2 t_2(\alpha t + 1) + \alpha_1 t_1(\alpha t + \alpha_2 t_2 + 1))} \tag{51}$$

$$x^* - y^* = \frac{(t_1 - t_2)(\alpha t + 1) - \alpha_1 t_1(t + t_2 - 1)}{3\alpha_2 t_2(\alpha t + 1) + 3\alpha_1 t_1(\alpha t + \alpha_2 t_2 + 1)} \tag{52}$$

Given the above closed-form equilibrium solutions, it is indeed possible to conduct welfare comparisons and comparative statics analysis as we have done in our previous model setups. Since our main objective for this section is merely to provide the analytical framework for this market structure, such extended analysis is beyond the scope of this current paper.

7.2 Duopoly in AB and Duopoly in BC

We then consider the market structure where firm 1 and firm 2 compete for route AB and BC separately by setting entrance fees p_1, p_2 for AB and q_1, q_2 for route BC, respectively. The setting of this hybrid structure can be illustrated by Figure 6.

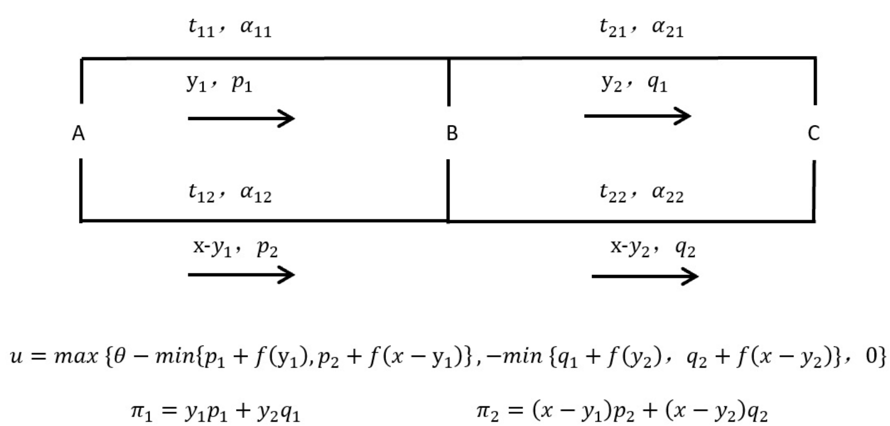


Figure 6: Duopoly in AB and Duopoly in BC

In equilibrium, suppose the total traffic is denoted by x while y_1 of them choose firm 1 in route AB, and y_2 of them choose firm 1 at route BC. Thus the equilibrium condition requires the following three equations.

$$\begin{aligned}
p_1 + t_{11}(1+\alpha_{11}y_1) &= p_2 + t_{12}(1+\alpha_{12}(x-y_1)) \\
q_1 + t_{21}(1+\alpha_{21}y_2) &= q_2 + t_{22}(1+\alpha_{22}(x-y_2)) \\
p_1 + t_{11}(1+\alpha_{11}y_1) + q_1 + t_{21}(1+\alpha_{21}y_2) &= 1-x
\end{aligned}$$

We can solve for the traffic flows

$$y_1 = \frac{t_{12}\alpha_{12}x + p_2 + t_{12} - p_1 - t_{11}}{t_{11}\alpha_{11} + t_{12}\alpha_{12}} \tag{53}$$

$$x - y_1 = \frac{t_{11}\alpha_{11}x + p_1 + t_{11} - p_2 - t_{12}}{t_{11}\alpha_{11} + t_{12}\alpha_{12}} \tag{54}$$

$$y_2 = \frac{t_{22}\alpha_{22}x + q_2 + t_{22} - q_1 - t_{21}}{t_{21}\alpha_{21} + t_{22}\alpha_{22}} \tag{55}$$

$$x - y_2 = \frac{t_{21}\alpha_{21}x + q_1 + t_{21} - q_2 - t_{22}}{t_{21}\alpha_{21} + t_{22}\alpha_{22}} \tag{56}$$

$$x = \frac{1 - \frac{t_{12}\alpha_{12}(p_1+t_{11})}{t_{11}\alpha_{11}+t_{12}\alpha_{12}} - \frac{t_{11}\alpha_{11}(p_2+t_{12})}{t_{11}\alpha_{11}+t_{12}\alpha_{12}} - \frac{t_{22}\alpha_{22}(q_1+t_{21})}{t_{21}\alpha_{21}+t_{22}\alpha_{22}} - \frac{t_{21}\alpha_{21}(q_2+t_{22})}{t_{21}\alpha_{21}+t_{22}\alpha_{22}}}{1 + \frac{t_{11}\alpha_{11}t_{12}\alpha_{12}}{t_{11}\alpha_{11}+t_{12}\alpha_{12}} + \frac{t_{21}\alpha_{21}t_{22}\alpha_{22}}{t_{21}\alpha_{21}+t_{22}\alpha_{22}}} \tag{57}$$

where x can be directly derived from Equation (36) as well.

Note that the profits for the two firms are given by

$$\Pi_1 = y_1p_1 + y_2q_1 \tag{58}$$

$$\Pi_2 = (x-y_1)p_2 + (x-y_2)q_2 \tag{59}$$

Substituting (53)-(56) into (58) and (59), we can have 4 first order conditions through the profit maximization process, which uniquely determine the equilibrium prices p_1, p_2, q_1 and q_2. Given the equilibrium prices, we can easily solve for the equilibrium traffic flows, completely characterized by y_1^*, y_2^* and x^*. The detailed expressions are offered in the Appendix.

8 Conclusion

Travel routes consisting of more than one segment are common for many commuters and travelers, and a relevant transportation policy question is how to manage such routes in terms of the market structure. Our analysis makes progress in understanding this issue in a multi-segment transportation structure of either a serial or a parallel nature.

The results show that in terms of traffic flows, neither monopoly nor duopoly competition can reach the socially optimal traffic flows. However if maximal traffic flows are a desired policy target, monopoly is the better performing market structure under the serial structure, while duopoly is preferable under the parallel structure.

It may be the case that transportation policy-makers are less concerned about traffic volumes compared with traditional notions of social welfare. We demonstrate that if two firms are symmetric in price competition, commuters are always better off under parallel competition than serial competition, when commuter utility incorporates price and time costs additively. Indeed for social welfare as a whole, parallel duopoly also comes out ahead of the serial duopoly arrangement. The only source of disparity in desired duopoly structure is on occasion, from the perspective of the firms. Firms' profits can be higher under the serial duopoly if the route is sufficiently unburdened in terms of travel times or route conditions. The opposing comparative firm profits and social welfare could potentially present a conflict of interest between firms in the market and governments and commuters for relatively less-traveled routes.

In terms of the welfare advantages of the parallel structure, through extensions of the model we find that the result is robust when taking into account firms' operating costs that depend on exogenous route features. The welfare advantage of the parallel structure is also robust to a generalization of an n-firm oligopoly competing over several parallel or serial segments. We also consider hybrid market structures of the baseline model, in which either both monopoly and duopoly market structures are present in the transport system, or both serial and parallel features are present.

We can see several directions for future research. For the welfare analysis, our model has assumed symmetric roles of the duopolists, while in practice the duopolists could be heterogeneous in terms of the features of the route they are managing. Future work could evaluate welfare under differing assumptions about the symmetry of the competing firms. In addition, all firms in our model have been profit maximizing in their objectives. A reality is that in many transport networks, some segments of the route are run by fully or partially state-owned organizations whose objectives may differ from those of private firms. A possible extension is to incorporate a mixed oligopoly structure (as for example in Li, Lien and Zheng, 2018[15]) into the transportation system using the framework studied here. Another note is that our model is general and does not attempt to make policy recommendations based on any actual statistics or data. Actual parameters could potentially be applied to the model to obtain quantitative welfare and surplus estimates. Finally, future work may apply a similar framework to more complex transportation networks than the ones we consider here.

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Appendix

A Proof of Proposition 2.2

First, we derive the results of comparative statics analysis with respect to parameters t and α.

$$\frac{\partial x_{Duo}^{S}}{\partial t} = -\frac{\alpha+1}{3(\alpha t+1)^2} < 0$$

$$\frac{\partial x_{Opt}^{S}}{\partial t} = -\frac{2\alpha+1}{(2\alpha t+1)^2} < 0$$

$$\frac{\partial x_{UE}^{S}}{\partial t} = -\frac{\alpha+1}{(\alpha t+1)^2} < 0$$

$$\frac{\partial x_{Mon}^{S}}{\partial t} = -\frac{\alpha+1}{2(\alpha t+1)^2} < 0$$

$$\frac{\partial x_{Duo}^{S}}{\partial \alpha} = -\frac{t(1-t)}{3(\alpha t+1)^2} < 0$$

$$\frac{\partial x_{Opt}^{S}}{\partial \alpha} = -\frac{2t(1-t)}{(2\alpha t+1)^2} < 0$$

$$\frac{\partial x_{UE}^{S}}{\partial \alpha} = -\frac{t(1-t)}{(\alpha t+1)^2} < 0$$

$$\frac{\partial x_{Mon}^{S}}{\partial \alpha} = -\frac{t(1-t)}{2(\alpha t+1)^2} < 0$$

Then, we derive the results of comparative statics analysis with respect to parameters t_i and α_i, $i = 1, 2$

$$\frac{\partial x_{Duo}^{S}}{\partial t_i} = -\frac{\alpha_1 t_1 + \alpha_2 t_2 + 1 + \alpha_i(1-t_1-t_2)}{3(\alpha_1 t_1 + \alpha_2 t_2 + 1)^2} < 0$$

$$\frac{\partial x_{Opt}^{S}}{\partial t_i} = -\frac{2\alpha_1 t_1 + 2\alpha_2 t_2 + 1 + 2\alpha_i(1-t_1-t_2)}{(2\alpha_1 t_1 + 2\alpha_2 t_2 + 1)^2} < 0$$

$$\frac{\partial x_{UE}^{S}}{\partial t_i} = -\frac{\alpha_1 t_1 + \alpha_2 t_2 + 1 + \alpha_i(1-t_1-t_2)}{(\alpha_1 t_1 + \alpha_2 t_2 + 1)^2} < 0$$

$$\frac{\partial x_{Mon}^{S}}{\partial t_i} = -\frac{\alpha_1 t_1 + \alpha_2 t_2 + 1 + \alpha_i(1-t_1-t_2)}{2(\alpha_1 t_1 + \alpha_2 t_2 + 1)^2} < 0$$

$$\frac{\partial x_{Duo}^{S}}{\partial \alpha_i} = -\frac{t_i(1-t_1-t_2)}{3(\alpha_1 t_1 + \alpha_2 t_2 + 1)^2} < 0$$

$$\frac{\partial x_{Opt}^{S}}{\partial \alpha_i} = -\frac{2t_i(1-t_1-t_2)}{(2\alpha_1 t_1 + 2\alpha_2 t_2 + 1)^2} < 0$$

$$\frac{\partial x_{UE}^S}{\partial \alpha_i} = -\frac{t_i(1-t_1-t_2)}{(\alpha_1 t_1+\alpha_2 t_2+1)^2} < 0$$

$$\frac{\partial x_{Mon}^S}{\partial \alpha_i} = -\frac{t_i(1-t_1-t_2)}{2(\alpha_1 t_1+\alpha_2 t_2+1)^2} < 0$$

B Proof of Proposition 3.2

First, we derive the comparative statics results with respect to parameters t_i, $i = 1, 2$ for the Duopoly scenario.

$$\begin{aligned}
&\frac{\partial x_{Duo}^P}{\partial t_1} \text{ (ignoring the denominator)}\\
=& -\alpha_2 t_2^2(2\alpha_2(\alpha_2 t_2+1)(4\alpha_2 t_2+3)+\alpha_1(\alpha_2(2t_2(2\alpha_2((2\alpha_2+1)t_2+4)+3)+9)+3))\\
&-2\alpha_1\alpha_2 t_1 t_2(\alpha_2 t_2+1)(8(\alpha_1+1)\alpha_2^2 t_2^2+2\alpha_2 t_2(2\alpha_1(t_2+3)+5)+2\alpha_1(t_2+3)+3)\\
&-\alpha_1^2 t_1^2(\alpha_2 t_2+1)^2(8(\alpha_1+1)\alpha_2^2 t_2^2+2\alpha_2 t_2(2\alpha_1(t_2+2)+3)-2\alpha_1(t_2-3)+3) < 0\\
&\frac{\partial x_{Duo}^P}{\partial t_2} \text{ (ignoring the denominator)}\\
=& -4\alpha_1^3(\alpha_2+2\alpha_1(\alpha_2+1))t_1^4(\alpha_2 t_2+1)^2\\
&-2\alpha_1^2 t_1^3(\alpha_2 t_2+1)(3\alpha_2(\alpha_2 t_2+1)+\alpha_1(\alpha_2((12\alpha_2+11)t_2+8)+7))\\
&-\alpha_1 t_1^2(\alpha_2(4\alpha_2 t_2+3)+\alpha_1(\alpha_2(t_2(\alpha_2((30\alpha_2+23)t_2+36)+26)+9)+6))\\
&-2\alpha_2 t_2 t_1(\alpha_1(2\alpha_2((5\alpha_2+3)t_2+3)+3)-\alpha_2^2 t_2)-3\alpha_2^2(2\alpha_2+1)t_2^2 < 0
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial x_{Duo}^P}{\partial \alpha_1}\\
=&\frac{\alpha_2 t_2(t_2(\alpha_2(t_1-1)-1)+t_1)}{(\alpha_1 t_1+\alpha_2 t_2(\alpha_1 t_1+1))^2}+\frac{2(2\alpha_2 t_2+3)(t_2(2\alpha_2(t_1-1)+1)+2t_1-3)}{(4\alpha_1 t_1+4\alpha_2 t_2(\alpha_1 t_1+1)+3)^2}\\
=&-\frac{\alpha_2 t_2((\alpha_2+1)t_2-t_1(\alpha_2 t_2+1))}{(\alpha_1 t_1+\alpha_2 t_2(\alpha_1 t_1+1))^2}-\frac{2(2\alpha_2 t_2+3)(3(1-t_2)+2((\alpha_2+1)t_2-t_1(\alpha_2 t_2+1)))}{(4\alpha_1 t_1+4\alpha_2 t_2(\alpha_1 t_1+1)+3)^2} < 0
\end{aligned}$$

where the last inequality holds by the fact that $t_2 < 1$ and $(\alpha_2+1)t_2 - t_1(\alpha_2 t_2+1) > 0$ where the latter is implied by equation (33).

$$\begin{aligned}
&\frac{\partial x_{Duo}^P}{\partial \alpha_2}\\
=&\frac{\alpha_1 t_1(t_1(\alpha_1(t_2-1)-1)+t_2)}{(\alpha_2 t_2+\alpha_1 t_1(\alpha_2 t_2+1))^2}+\frac{2(2\alpha_1 t_1+3)(t_1(2\alpha_1(t_2-1)+1)+2t_2-3)}{(4\alpha_2 t_2+4\alpha_1 t_1(\alpha_2 t_2+1)+3)^2}\\
=&-\frac{\alpha_1 t_1((\alpha_1+1)t_1-t_2(\alpha_1 t_1+1))}{(\alpha_2 t_2+\alpha_1 t_1(\alpha_2 t_2+1))^2}-\frac{2(2\alpha_1 t_1+3)(3(1-t_1)+2((\alpha_1+1)t_1-t_2(\alpha_1 t_1+1)))}{(4\alpha_2 t_2+4\alpha_1 t_1(\alpha_2 t_2+1)+3)^2} < 0
\end{aligned}$$

where the last inequality holds by the fact that $t_1 < 1$ and $(\alpha_1+1)t_1 - t_2(\alpha_1 t_1+1) > 0$ where the latter is implied by equation (34).

Second, we show the results fro the Social Optimum scenario. For $\frac{\partial x_{Opt}^P}{\partial t_i} < 0$ $(i = 1, 2)$, the proof is by contradiction. Suppose there exist t_i, α_i $(i = 1, 2)$ and i^* such that x_{Opt}^P increases when t_{i^*} increases. Note that the first order conditions imply that the equilibrium travel time for both routes should decrease since $f_i(x_i) = 1 - x_{Opt}^P$ for $i = 1, 2$. Since $f_i(x_i)$ is increasing in x_i for $i = 1, 2$, a decrease in f_{-i^*} implies a decrease in x_{-i^*}. Since $x_{Opt}^P = x_{i^*} + x_{-i^*}$, this must mean x_{i^*} increases. The increase in x_{i^*} together with the increase in t_{i^*} determines the increase in $f_{i^*}(x_{i^*})$ based on the fact that f_{i^*} is increasing in x_{i^*} and t_{i^*}. However, the increase in f_{i^*} contradicts with the decrease in f_{-i^*} in equilibrium. We can prove $\frac{\partial x_{Opt}^P}{\partial \alpha_i} < 0$ $(i = 1, 2)$ by applying the same logic.

Using an analogous reasoning approach, we can also easily prove the results for the Price-free scenario ($\frac{\partial x_{UE}^P}{\partial t_i} < 0$ and $\frac{\partial x_{UE}^P}{\partial \alpha_i} < 0$) and for the Monopoly scenario ($\frac{\partial x_{Mon}^P}{\partial \alpha_i} < 0$) where one extra step relying on the fact that equilibrium price is independent of α_i is needed.

Last, to show $\frac{\partial x_{Mon}^P}{\partial t_i} < 0$ for the Monopoly scenario, note

$$\begin{aligned}
\frac{\partial x_{Mon}^P}{\partial t_i} &= \frac{(\alpha_i - t_{-i}(\alpha_1 + \alpha_2))(t_1\alpha_1 + t_2\alpha_2 + t_1\alpha_2 t_2\alpha_2) - (\alpha_i + t_{-i}\alpha_1\alpha_2)(t_1\alpha_1 + t_2\alpha_2 - t_1 t_2(\alpha_1 + \alpha_2))}{2(t_1\alpha_1 + t_2\alpha_2 + t_1\alpha_2 t_2\alpha_2)^2} \\
&= \frac{(-t_{-i}t_1\alpha_1 - t_{-i}t_2\alpha_2 + \alpha_i t_1 t_2)(\alpha_1 + \alpha_2 + \alpha_1\alpha_2)}{2(t_1\alpha_1 + t_2\alpha_2 + t_1\alpha_2 t_2\alpha_2)^2} \\
&= \frac{-t_{-i}^2\alpha_{-i}(\alpha_1 + \alpha_2 + \alpha_1\alpha_2)}{2(t_1\alpha_1 + t_2\alpha_2 + t_1\alpha_2 t_2\alpha_2)^2} < 0
\end{aligned}$$

C Circuits and Thévenin's Theorem

C.1 Analogy with Equilibria in Transportation Networks and Circuits

This analogy is based on one key assumption: the latency for any specific route in the transportation network has a linear form, $f(x) = t(1 + \alpha x)$, where x denotes the traffic flow through the route.

The analogy can be described as follows:

1. Regard the traffic flow in the transportation network as the current in one circuit;
2. Regard the cost (including latency and entrance fee) in one specific route AB as the electrical potential difference between two nodes in the circuit;[5]
3. Regard the total cost from A to C (through some mediators) as the electrical potential difference between A and C;[6]

[5] Then the linear latency function of the route can be regarded as a wire connecting A and B that includes a $U = p + t$ voltage source in series connection with a resistance of $R = t\alpha$.

[6] When traffic network is in equilibrium, multiple routes from A to C must have the same cost, which coincides with Kirchhoff's law in circuit.

4. Regard the heterogeneous consumer as a $1V$ voltage source with 1Ω internal resistance.[7]

C.2 Thévenin's Theorem

Thévenin's theorem holds that:[8]

- Any linear electrical network with voltage and current sources and resistances only can be replaced at terminals A-B by an equivalent voltage source V in series connection with an equivalent resistance R.
- The equivalent voltage V is the voltage obtained at terminals A-B of the network with terminals A-B open circuited.
- The equivalent resistance R is the resistance that the circuit between terminals A and B would have, if all ideal voltage sources in the circuit were replaced by a short circuit and all ideal current sources were replaced by an open circuit.

In circuit theory terms, the theorem allows for any one-port network with any complex structure to be reduced to a single voltage source and a single impedance. The Thévenin's equivalences of serial and parallel structures mentioned in Section 4.1 are thus special case application of the general theorem.

D Oligopoly with n firms: Analysis Details

D.1 Calculation Process for Parallel Structure

Since n routes are identical, without loss of generality, we analyze the symmetric equilibrium in which all firms set the same price $p_i = p^*$. The equilibrium requires the following: if all other $n-1$ firms set the price at p^*, it is optimal for the last firm to set the price at p^*. Let $\Pi(x, y)$ denote the profit when the last firm sets the price at x while other firms set the price at y. Therefore, the following condition holds,

$$\left.\frac{\partial \Pi(p, p^*)}{\partial p}\right|_{p=p^*} = 0$$

We first calculate the flow of each route when the last firm sets the price at p while other firms set the price at p^*. Assume the flow of the last route equals x while the flow of all other routes is identical and equals to $\bar{x}$. Then, the total flow equals $(n-1)\bar{x} + x$. The commuter with value

[7]When traffic flow is x, it means that consumer with valuation $1-x$ has zero utility. This zero utility condition coincides with Kirchhoff's law and the corresponding external voltage should be $1-x$. This external voltage source is equivalent to a $1V$ source with internal resistance 1Ω.

[8]From Wikipedia, https://en.wikipedia.org/wiki/Th%C3%A9venin%27s_theorem

$1-(n-1)\bar{x}-x$ is indifferent among three options: (1) not traveling, (2) choosing the last route and (3) choosing other routes. Mathematically, the following two equations will determine the flow parameters x and$\bar{x}$ given p and p^*,

$$\begin{aligned} p+t(1+n\alpha x) &= p^*+t(1+n\alpha\bar{x}) \\ 1-(n-1)\bar{x}-x &= p^*+t(1+n\alpha\bar{x}) \end{aligned}$$

Rearranging the first equation, we get

$$x=\bar{x}+\frac{p^*-p}{n\alpha t}$$

Substituting x in the second equation, we obtain

$$\begin{aligned} \bar{x}(p,p^*) &= \frac{1}{n(1+\alpha t)}(1-p^*-t+\frac{p^*-p}{n\alpha t}) \\ x(p,p^*) &= \frac{1}{n(1+\alpha t)}(1-p^*-t+\frac{p^*-p}{n\alpha t})+\frac{p^*-p}{n\alpha t} \end{aligned}$$

The profit for the last firm is defined by

$$\Pi(p,p^*)=px(p,p^*)=\frac{p(1-p^*-t+\frac{p^*-p}{n\alpha t})}{n(1+\alpha t)}+\frac{p^*p-p^2}{n\alpha t}$$

with first order condition

$$\frac{1-p^*-t+\frac{p^*-2p}{n\alpha t}}{n(1+\alpha t)}+\frac{p^*-2p}{n\alpha t}=0$$

where $p=p^*$.

Price We solve for p^* and get

$$p^*=\frac{n\alpha t(1-t)}{2n\alpha t+n-1} \tag{60}$$

Flow The equilibrium flow of each route is thus

$$x_i^*=\frac{1-p^*-t}{n(1+\alpha t)}=\frac{(1-t)(n\alpha t+n-1)}{n(1+\alpha t)(2n\alpha t+n-1)} \tag{61}$$

and total flow is

$$X=\frac{(1-t)(n\alpha t+n-1)}{(1+\alpha t)(2n\alpha t+n-1)} \tag{62}$$

Profit The profit for each firm is hence

$$\Pi_i=p^*x_i^*=\frac{\alpha t(1-t)^2(n\alpha t+n-1)}{(1+\alpha t)(2n\alpha t+n-1)^2} \tag{63}$$

Consumer Surplus We first calculate the average surplus for the consumer, which is exactly $\frac{X}{2}$. Therefore, the consumer surplus is defined by

$$\frac{X^2}{2} = \frac{(1-t)^2(n\alpha t + n - 1)^2}{2(1+\alpha t)^2(2n\alpha t + n - 1)^2} \tag{64}$$

Social Surplus The total social surplus takes the summation of firms' profits and consumer surplus,

$$\begin{aligned}
\text{Social Surplus} &= \text{Total Profit} + \text{Consumer Surplus} \\
&= n \times \text{Firm Profit} + \text{Consumer Surplus} \\
&= n\frac{\alpha t(1-t)^2(n\alpha t + n - 1)}{(1+\alpha t)(2n\alpha t + n - 1)^2} + \frac{(1-t)^2(n\alpha t + n - 1)^2}{2(1+\alpha t)^2(2n\alpha t + n - 1)^2} \\
&= \frac{(1-t)^2(n\alpha t + n - 1)^2}{2(1+\alpha t)^2(2n\alpha t + n - 1)^2}[2(1+\alpha t)(n\alpha t) + (n\alpha t + n - 1)] \\
&= \frac{(1-t)^2(n\alpha t + n - 1)^2(2n\alpha^2 t^2 + 3n\alpha t + n - 1)}{2(1+\alpha t)^2(2n\alpha t + n - 1)^2}
\end{aligned} \tag{65}$$

D.2 Comparison Between Serial and Parallel Structures

D.2.1 Total Flow

$$\frac{X^P_{Oli}}{X^S_{Oli}} = \frac{(n+1)(n\alpha t + n - 1)}{(2n\alpha t + n - 1)} > \frac{n+1}{2} > 1$$

Hence, $X^P_{Oli} > X^S_{Oli}$.

D.2.2 Profit

$$\begin{aligned}
\frac{\Pi^P_{Oli}}{\Pi^S_{Oli}} - 1 &= \frac{(n+1)^2 t\alpha(n\alpha t + n - 1) - (2n\alpha t + n - 1)^2}{(2n\alpha t + n - 1)^2} \\
&= \frac{(n-1)^2 n(\alpha t)^2 + (n-1)^3(\alpha t) - (n-1)^2}{(2n\alpha t + n - 1)^2} \\
&= \frac{(n-1)^2(n\alpha t - 1)(\alpha t + 1)}{(2n\alpha t + n - 1)^2}
\end{aligned}$$

Hence, $\Pi^P_{Oli} > \Pi^S_{Oli}$ if and only if $\alpha t > \frac{1}{n}$.

D.2.3 Consumer Surplus

$$\frac{CS^P_{Oli}}{CS^S_{Oli}} = \frac{(n+1)^2(n\alpha t + n - 1)^2}{(2n\alpha t + n - 1)^2} > \frac{(n+1)^2}{4} > 1$$

Hence, $CS^P_{Oli} > CS^S_{Oli}$.

D.2.4 Total Surplus

When $n \geq 3$,

$$\begin{aligned}
\frac{TS^P_{Oli}}{TS^S_{Oli}} &= \frac{(n+1)^2(n\alpha t+n-1)(2n\alpha^2t^2+3n\alpha t+n-1)}{(2n\alpha t+n-1)^2(2n\alpha t+2n+1)} \\
&> \frac{(n+1)^2(2n\alpha^2t^2+3n\alpha t+n-1)}{2(2n\alpha t+n-1)(2n\alpha t+2n+1)} \\
&= \frac{2n(n+1)^2\alpha^2t^2+3n(n+1)^2\alpha t+(n+1)^2(n-1)}{8n(\alpha t)^2+12n^2\alpha t+(4n+2)(n-1)}
\end{aligned}$$

$$\begin{aligned}
2n(n+1)^2 &> 8n \\
3n(n+1)^2 &> 12n^2 \\
(n+1)^2(n-1) &> (4n+2)(n-1)
\end{aligned}$$

Hence, $TS^P_{Oli} > TS^S_{Oli}$.

For the special case when $n = 2$,

$$\begin{aligned}
\frac{TS^P_{Oli}}{TS^S_{Oli}} &= \frac{9(2\alpha t+1)(4\alpha^2t^2+6\alpha t+1)}{(4\alpha t+1)^2(4\alpha t+5)} \\
&= \frac{72(\alpha t)^3+144(\alpha t)^2+72(\alpha t)+9}{64(\alpha t)^3+112(\alpha t)^2+44(\alpha t)+5} \\
&> 1
\end{aligned}$$

E Extension III-2: Duopoly in AB and Duopoly in BC

We provide the close-form equilibrium solutions for the hybrid structure where 2 firms compete parallelly both over the AB segment and the BC segment. As shown below, the mathematical expressions for traffic flows are very lengthy, and we decide not to conduct comparative statics analysis or welfare comparisons.

The traffic flow on firm 1's route over the AB segment is

$$
\begin{aligned}
y_1^* = & -6\alpha_{11}t_{11}^2(\alpha_{12}t_{12} + \alpha_{22}t_{22} + 1)(\alpha_{22}t_{22}(\alpha_{12}t_{12} + 1) + \alpha_{21}t_{21}(\alpha_{12}t_{12} + \alpha_{22}t_{22} + 1)) \\
& + t_{11}(-2\alpha_{21}(\alpha_{11} + 2\alpha_{21})t_{21}^2(\alpha_{12}t_{12} + \alpha_{22}t_{22} + 1)^2 - t_{21}(\alpha_{21}(6\alpha_{12}t_{12} \\
& + \alpha_{11}(t_{12}(\alpha_{12}(4t_{22} - 6) - 3) + t_{22} - 3) + 3) + 2\alpha_{22}t_{22}(\alpha_{21}(3\alpha_{12}t_{12} + 4) \\
& + \alpha_{11}(3\alpha_{12}t_{12} - \alpha_{21}(2t_{12} + 1) + 1)))(\alpha_{12}t_{12} + \alpha_{22}t_{22} + 1) \\
& + \alpha_{22}t_{22}((\alpha_{12}t_{12} + 1)(\alpha_{11}(3t_{12} - t_{22} + 3) + 6(\alpha_{11} - 1)\alpha_{12}t_{12} - 3) \\
& + 2\alpha_{22}t_{22}(\alpha_{11}(2t_{12} + 1) + 3(\alpha_{11} - 1)\alpha_{12}t_{12} - 2))) \\
& + t_{12}(-2\alpha_{22}^2t_{22}^2(3\alpha_{12}(t_{21} - 1) - 2\alpha_{21}t_{21} - 2)(\alpha_{21}t_{21} + 1) \\
& + \alpha_{22}t_{22}(4\alpha_{21}^2t_{21}^2(\alpha_{12}(t_{12} - t_{22} + 1) + 2) + \alpha_{21}t_{21}(\alpha_{12}(t_{12}(7 - 6\alpha_{12}(t_{21} - 1)) - 8t_{21} - 5t_{22} \\
& + 13) + 11) + \alpha_{12}(t_{12}(3 - 6\alpha_{12}(t_{21} - 1)) - 5t_{21} - t_{22} + 6) + 3) \\
& - \alpha_{21}t_{21}(\alpha_{12}t_{12} + 1)(2\alpha_{12}(t_{21} + 2t_{22} - 3) + 4\alpha_{21}t_{21}(\alpha_{12}(t_{22} - 1) - 1) - 3))/ \\
& 3(4\alpha_{21}t_{21} + 4\alpha_{12}t_{12}(\alpha_{21}t_{21} + 1) + 4\alpha_{22}t_{22}(\alpha_{21}t_{21} + 1) \\
& + 4\alpha_{11}t_{11}(\alpha_{12}t_{12} + \alpha_{22}t_{22} + 1) + 3)(\alpha_{12}t_{12}(\alpha_{22}t_{22} + \alpha_{21}t_{21}(\alpha_{22}t_{22} + 1)) \\
& + \alpha_{11}t_{11}(\alpha_{22}t_{22}(\alpha_{12}t_{12} + 1) + \alpha_{21}t_{21}(\alpha_{12}t_{12} + \alpha_{22}t_{22} + 1)))
\end{aligned}
$$

The traffic flow on firm 1's route over the BC segment is

$$
\begin{aligned}
y_2^* = & -2\alpha_{11}t_{11}^2(\alpha_{12}t_{12} + \alpha_{22}t_{22} + 1)((\alpha_{12}t_{12} + 1)((2\alpha_{11} + \alpha_{21})t_{21} - 2\alpha_{11}t_{22}) \\
& + \alpha_{22}t_{22}(2\alpha_{11}(t_{12} + t_{21} - 1) + 3\alpha_{12}t_{12} + \alpha_{21}t_{21} + 1)) + t_{11}(-6\alpha_{11}\alpha_{21}t_{21}^2(\alpha_{12}t_{12} + \alpha_{22}t_{22} + 1)^2 \\
& - t_{21}(2\alpha_{12}\alpha_{21}t_{12}(3\alpha_{22}t_{22} + 1) + \alpha_{11}(-3\alpha_{21}(t_{22} + 1) - 6(\alpha_{21} - 1)\alpha_{22}t_{22} + t_{12}(\alpha_{21} - 2\alpha_{12}(\alpha_{21}(2t_{22} + 1) - 4) \\
& + 2(3\alpha_{12} + 2\alpha_{21})\alpha_{22}t_{22}) + 3))(\alpha_{12}t_{12} + \alpha_{22}t_{22} + 1) + t_{22}(\alpha_{11}(8\alpha_{12}^2t_{12}^2 + 11\alpha_{12}t_{12} + 2\alpha_{22}^2t_{22}((3\alpha_{12} - 2)t_{12} + 3) \\
& + \alpha_{22}(t_{12}(\alpha_{12}((6\alpha_{12} - 5)t_{12} + 7t_{22} + 13) - 4) + 3(t_{22} + 2)) + 3) - \alpha_{12}\alpha_{22}t_{12}(6\alpha_{12}t_{12} + 6\alpha_{22}t_{22} + 5))) \\
& + \alpha_{12}t_{12}(-6\alpha_{21}t_{21}^2(\alpha_{22}t_{22} + 1)(\alpha_{12}t_{12} + \alpha_{22}t_{22} + 1) + t_{21}(3(\alpha_{22}t_{22} + 1)(\alpha_{21}(t_{22} + 1) + 2(\alpha_{21} - 1)\alpha_{22}t_{22} - 1) \\
& + t_{12}(2\alpha_{12}(\alpha_{21}(2t_{22} + 1) + 3(\alpha_{21} - 1)\alpha_{22}t_{22} - 2) - \alpha_{21}(\alpha_{22}t_{22} + 1))) + t_{22}(3(2\alpha_{22} + 1)(\alpha_{22}t_{22} + 1) \\
& + (4\alpha_{12} + (6\alpha_{12} - 1)\alpha_{22})t_{12}))/3(4\alpha_{21}t_{21} + 4\alpha_{12}t_{12}(\alpha_{21}t_{21} + 1) + 4\alpha_{22}t_{22}(\alpha_{21}t_{21} + 1) + 4\alpha_{11}t_{11} \\
& (\alpha_{12}t_{12} + \alpha_{22}t_{22} + 1) + 3)(\alpha_{12}t_{12}(\alpha_{22}t_{22} + \alpha_{21}t_{21}(\alpha_{22}t_{22} + 1)) + \alpha_{11}t_{11}(\alpha_{22}t_{22}(\alpha_{12}t_{12} + 1) \\
& + \alpha_{21}t_{21}(\alpha_{12}t_{12} + \alpha_{22}t_{22} + 1)))
\end{aligned}
$$

The total traffic flow is

$$
\begin{aligned}
x^* = &-\alpha_{11}t_{11}^2(\alpha_{22}t_{22}((\alpha_{12}t_{12}+1)(2\alpha_{11}(3t_{12}+t_{22}-3)+6\alpha_{12}t_{12}+3)+2\alpha_{22}t_{22}(2\alpha_{11}(t_{12}-1)+3\alpha_{12}t_{12}+1))\\
&+t_{21}(\alpha_{12}t_{12}+\alpha_{22}t_{22}+1)(3\alpha_{21}(2\alpha_{11}(t_{12}+t_{22}-1)+2\alpha_{12}t_{12}+1)+2(2\alpha_{11}+\alpha_{21})\alpha_{22}t_{22}))\\
&-t_{11}(\alpha_{21}t_{21}^2(\alpha_{12}t_{12}+\alpha_{22}t_{22}+1)(4\alpha_{12}\alpha_{21}t_{12}+\alpha_{11}(-6\alpha_{21}+2(\alpha_{12}+3\alpha_{21})t_{12}\\
&+6(\alpha_{21}+\alpha_{22})t_{22}+3))+t_{21}(\alpha_{11}(2\alpha_{22}^2t_{22}^2(3\alpha_{12}t_{12}+\alpha_{21}(2t_{12}-3)+3)+\alpha_{22}t_{22}(6\alpha_{12}^2t_{12}^2+13\alpha_{12}t_{12}\\
&+\alpha_{21}(t_{12}(8\alpha_{12}(t_{12}+t_{22}-2)+13)+9(t_{22}-2))+6)+\alpha_{21}(t_{12}(\alpha_{12}(t_{12}(\alpha_{12}(4t_{22}-6)+9)+13t_{22}-18)+6)\\
&+6t_{22}-9))+\alpha_{12}\alpha_{21}t_{12}(6\alpha_{12}t_{12}(\alpha_{22}t_{22}+1)+(2\alpha_{22}t_{22}+3)(3\alpha_{22}t_{22}+2)))+\alpha_{22}t_{22}(6\alpha_{12}t_{12}(\alpha_{12}t_{12}\\
&+\alpha_{22}t_{22}+1)+\alpha_{11}(t_{12}(\alpha_{12}((9-6\alpha_{12})t_{12}+5t_{22}-18)+6)-2\alpha_{22}t_{22}((3\alpha_{12}-2)t_{12}+3)\\
&+3(t_{22}-3))))-\alpha_{12}t_{12}(6\alpha_{22}^2(t_{21}-1)t_{22}^2(\alpha_{21}t_{21}+1)+\alpha_{22}t_{22}(t_{12}(\alpha_{21}t_{21}+1)(6\alpha_{12}(t_{21}-1)+2\alpha_{21}t_{21}+3)\\
&+3(t_{21}(\alpha_{21}(t_{21}(2\alpha_{21}(t_{22}-1)+3)+3(t_{22}-2))+2)+t_{22}-3))+\alpha_{21}t_{21}(t_{21}(6\alpha_{21}(t_{22}-1)+3)\\
&+t_{12}(2\alpha_{12}(t_{21}+2t_{22}-3)+2\alpha_{21}t_{21}(2\alpha_{12}(t_{22}-1)+1)+3)+6t_{22}-9))/3(4\alpha_{21}t_{21}+4\alpha_{12}t_{12}(\alpha_{21}t_{21}+1)\\
&+4\alpha_{22}t_{22}(\alpha_{21}t_{21}+1)+4\alpha_{11}t_{11}(\alpha_{12}t_{12}+\alpha_{22}t_{22}+1)+3)(\alpha_{12}t_{12}(\alpha_{22}t_{22}\\
&+\alpha_{21}t_{21}(\alpha_{22}t_{22}+1))+\alpha_{11}t_{11}(\alpha_{22}t_{22}(\alpha_{12}t_{12}+1)+\alpha_{21}t_{21}(\alpha_{12}t_{12}+\alpha_{22}t_{22}+1)))
\end{aligned}
$$

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