Monopoly Platform in Two-Sided Markets with Heterogeneous Agents and Limited Market Size[*]

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Abstract

We study a monopoly platform's profit optimization problem in a two-sided market with heterogeneous agents. Our setup departs from Armstrong (2006)'s monopoly model by assuming both (1) a continuum of agents of limited size on each side of the market and (2) heterogeneous utility of agents with Hotelling specification. We show that the monopoly's optimal pricing strategy always results in a corner solution in terms of the equilibrium market share, in contract to the interior solution result in literature. We also solve for the social planner's optimization problem and obtain a similar corner solution result. Furthermore, we conduct welfare comparison between the monopoly case and the socially optimal case, and characterize the conditions under which the monopoly outcome can be socially optimal.

Keywords: Network externalities; Monopoly Platform; Social optimum; Heterogeneous Agents; Two-sided Markets; Hotelling Model

JEL classification: L5, L82, L86, L96

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1. Introduction

With the rapid development of network technology, online platforms become more and more prevalent in societies' social and economic activities. Many regard the network platforms as a novel business pattern that brings profit to the operator and improves social welfare to the society.

Assuming the existence of network externalities between the two sides of a market, we study a monopoly platform's profit optimization problem. Departing from the monopoly platform model in Armstrong (2006), we assume that the total number of agents on each side of the market is normalized to one so that the market has a limited size. We also assume that agents have different transport costs to access the platform via Hotelling specification and they make decisions by comparing their utilities between joining and not joining the platform. This setup featuring limited market size, specified heterogeneity, and endogenous demand, allows us to provide a tractable framework for studying the behavior of the monopoly platform behavior and conducting welfare analysis according.

We show that the optimal strategy of the monopoly platform is never an interior solution in terms of the market shares. We fully characterize the optimal strategy of the monopoly platform and show how the result depends on the cost structure of the two-side market and the degree of the network externalities. When cost structures of the two sides are relatively similar, the optimal outcome is no coverage of service if the externalities are small and full coverage of service if the externalities are large. When cost structures of the two sides are dissimilar, no coverage, partial coverage, and full coverage of service can all be possible depending on the size of network externalities: In particular, if network externalities are of medium size, the optimal outcome has partial coverage of service for the relatively cost-inefficient side and full coverage of service for the relatively cost-efficient side.

We also solve for the social planner's optimization problem and show how the social optimal result depends on the cost structure of the two-side market and the degree of the network externalities. Comparing the results for the monopoly case and the social optimum case, we provide necessary and sufficient conditions under which the monopoly outcome is socially optimal. A symmetric environment is also considered as a special case of our general setup.

Our study belongs to the Industrial Organization literature on platforms with network externalities. Beginning with the analysis of single monopoly markets (Parker and van Alstyne 2000; Baye and Morgan 2001; Rochet and Tirole 2002; Schmalensee 2002; Armstrong, 2006), studies in this field have focused on platform pricing with different market structures (including the social planner's problem for proprietary platforms and associations); the competitiveness of private platforms under different scenarios (such as single-home/multi-home and homogeneous/differentiated products); competitive bottlenecks; exclusive contracts; and some other important factors such as alternative tariffs, market demand price elasticity, transaction volumes, and buyers' and sellers' net surplus.

Rochet and Tirole (2003) study the pricing strategy of "payment-card" type of platforms in two-sided markets, considering both the monopoly case and the duopoly case, with different objective functions of the platforms: either to maximize the profit or to maximize the social welfare. Rochet and Tirole (2006) further consider this type of platform model that integrates both usage and membership externalities and study obtain new results on the mix of usage and membership charges.

Mostly related to our work, Armstrong (2006) studies the pricing strategy of "shopping mall" type of platforms in two-sided markets, considering both the monopoly case and the duopoly case. For the monopoly case, the market size is not assumed to be limited and the demand function is assumed in a general form that is exogenously given, which differ from our setup.[1] Armstrong and Wright (2007) further study the duopoly case of competition between two platforms, and consider three cases: (1) product differentiation on both sides; (2) product differentiation on one side only; (3) no product differentiation on both side. For both models (1) and (2), agents' utilities are assumed high enough to always use at least one platform's service. For model (3), agents are assumed to be homogeneous and the only factor that matters is the network effect.

Literature on two-sided platforms also studies the pricing behavior of the seller side of the market, taking the platform as given (Li and Zheng, 2004; Li, Lien, and Zheng, 2017、2018; Lien, Mazalov, Melnik, and Zheng, 2016; Mazalov and Melnik, 2016; Mazalov, Chirkova, Zheng, and Lien, 2018). Among many others, Li, Lien, and Zheng (2017) consider the sequential competition between sellers in a two-sided software market, and Mazalov, Chirkova, Zheng, and Lien (2018) study the competition (in terms of optimal contracting strategy and pricing strategy) of the virtual operators (seller side) of a two-sided telecommunication market.

Studies in this literature have made substantial achievements and opened a new field of platform economy. Our study contributes to this literature by studying the platform's optimization problem under new environment (limited market size, heterogeneous agents, and endogenized demand[2]), and our results show that the optimal outcome is always a corner solution, regardless whether the object is to maximize platform profit or social welfare. Based on the welfare comparison outcome, we also provide necessary and sufficient conditions under which the monopoly is socially optimal. Thus, our work fills a gap in the literature with

[1] We conduct a more detailed comparison between Armstrong (2006) and our model in the next section after we introduce the model setup.

[2] By endogenous demand, we mean dropping the single-homing assumption in the literature (Caillaud and Julien 2003; Armstrong 2006). Instead of assuming that every agent must choose one platform, we allow an agent to choose not to join any platform if he or she receives less utility by joining compared to not joining the platform.

a solution to an optimization problem not studied by previous work.

The remainder of our paper proceeds as follows: Section 2 describes the model setup and highlights its difference from the existing models; In Section 3 we study the monopoly's problem and fully characterize the optimal solution; In Section 4 we consider the social planner's problem and solve for social optimum; Section 5 presents the welfare comparison results; Section 6 summarizes and discusses directions for future work. Detailed proofs are provided in the Appendix.

2. Model Setup

In a monopoly platform economy, there are two groups of agents: group 1 (also known as group of buyers) and group 2 (also known as group of sellers), with a continuum of mass 1 on each side. We suppose that both buyers and sellers are uniformly and independently located along a line of length 1, each of whom is identified by location x_1 ($0 \le x_1 \le 1$) and x_2 (($0 \le x_2 \le 1$)), respectively. Agents' utilities from group 1 and group 2 by using the platform located at x_0 ($0 \le x_0 \le 1$) are characterized by the following two expressions, respectively:

$$u_{x_1} = \alpha_1 n_2 - p_1 - |x_1 - x_0| t_1 ; \tag{1}$$

$$u_{x_2} = \alpha_2 n_1 - p_2 - |x_2 - x_0| t_2 ; \tag{2}$$

In the above expressions, the first term captures the utility from network externality of the other side of the market, the second term is the cost of price the agent has to pay to access the platform, and the third term is essentially the agent's heterogeneous transport cost of accessing the platform, via Hotelling speciation. To be more specific, $\alpha_i, i = 1, 2$ measure the degree of interactive benefit gained by a group i's member from the other side of market participants, $n_i, i = 1, 2$ is the size of group i, $p_i, i = 1, 2$ is the price paid to the platform by a group i's agent, and $t_i, i = 1, 2$ measures the per unit transport cost to access the platform by a group i's agent.

Note that our setup is different from both the Monopoly model and the Duopoly models in Armstrong (2006). In his Monopoly model, the demand function is assumed to have a general form, agents of the same group are assumed to have homogeneous utility, which corresponds to the sum of the first two terms in our utility function forms, and the market size is implicitly assumed to be unlimited. In contrast, by assuming a continuum of agents of mass 1 with heterogeneous transport cost to access the platform, we can derive demand function from the marginal agent condition, and solve for the optimal solution explicitly. In his Duopoly models, although the market size is assumed limited and Hotelling specification is adopted, any agent must join a platform and has no option of opting out, while in our setup, an agent decides whether to join the platform in order to maximize utility (and only if the net utilities are non-negative do the agents consider using a platform and became single-homing). The modelling differences among these three models are described in the table below.

Table 1 Comparison with Armstrong (2006)'s Monopoly Model and Duopoly Model

	Armstrong (2006)'s Monopoly Model	Armstrong (2006)'s Duopoly Model	Our Model
Agent Type	Homogeneous	Heterogeneous	Heterogeneous
Market Size	No limit	Limited	Limited
Demand Function	General	Hotelling Specification	Hotelling Specification
Freedom to Exit Market	Yes	No	Yes

Following the literature, we study two optimization problems with the objective to maximize platform profit or social welfare, respectively.

The monopoly platform's profit is:

$$\pi = n_1(p_1 - f_1) + n_2(p_2 - f_2), \tag{3}$$

where f_1 *and* f_2, assumed exogenously given, are the serving cost per agent for groups 1 and 2, respectively.

The social welfare is:

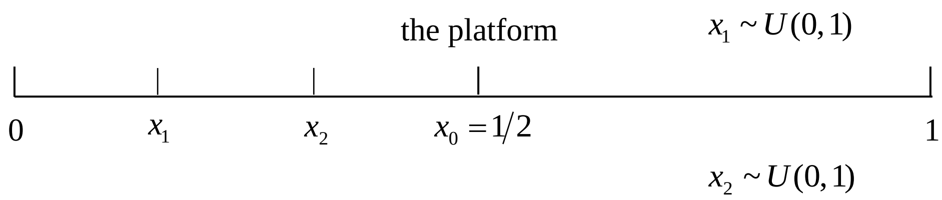
$$W = (\alpha_1 + \alpha_2)n_1 n_2 - n_1 f_1 - n_2 f_2 - \int_{u_{x_1} \geq 0, x_1 \in [0,1]} |x_1 - x_0| t_1 dx_1 - \int_{u_{x_2} \geq 0, x_2 \in [0,1]} |x_2 - x_0| t_2 dx_2 . \tag{4}$$

3. Monopoly

Our main goal is to solve the monopoly platform's profit maximization problem. The monopoly chooses its location x_0 and prices p_1, p_2 to maximize its profit, defined by (3).

First, it is straightforward to see that letting $x_0 = \frac{1}{2}$ is always an optimal choice for the platform. In other words, the middle point of the beach in the Hotelling model is an optimal location for the platform, as Figure 1 shows.

Figure 1 Optimal Location of the Monopoly Platform



Second, we derive the demand functions for both sides of the market. Since $x_0 = \frac{1}{2}$, by symmetry, we only need to consider agents located on the left-hand side of the platform ($x_i \leq \frac{1}{2}, i = 1, 2$). From $u_{x_1} = \alpha_1 n_2 - p_1 - (\frac{1}{2} - x_1)t_1 \geq 0$, we have $x_1 \geq \frac{1}{2} - \frac{1}{t_1}(\alpha_1 n_2 - p_1)$; thus, we obtain the demand function for group 1 (left) as $n_{1_{left}} = \frac{1}{t_1}(\alpha_1 n_2 - p_1)$ and that for group 2 (left) as $n_{2_{left}} = \frac{1}{t_2}(\alpha_2 n_1 - p_2)$. Therefore, the demand functions for these two groups respectively are:

$$n_1 = \begin{cases} 0 \quad if \quad \frac{2}{t_1}(\alpha_1 n_2 - p_1) < 0 \\ \frac{2}{t_1}(\alpha_1 n_2 - p_1) \quad if \quad \frac{2}{t_1}(\alpha_1 n_2 - p_1) \in [0,1] \\ 1 \quad if \quad \frac{2}{t_1}(\alpha_1 n_2 - p_1) > 1 \end{cases}; \tag{5}$$

$$n_2 = \begin{cases} 0 \quad if \quad \frac{2}{t_2}(\alpha_2 n_1 - p_2) < 0 \\ \frac{2}{t_2}(\alpha_2 n_1 - p_2) \quad if \quad \frac{2}{t_2}(\alpha_2 n_1 - p_2) \in [0,1] \\ 1 \quad if \quad \frac{2}{t_2}(\alpha_2 n_1 - p_2) > 1 \end{cases}. \tag{6}$$

With the demand functions available, we will be able to solve for the optimization problem for the monopoly platform. In general, one will think that profit maximization can potentially lead to an interior solution where $0 < n_1^* < 1, 0 < n_2^* < 1$. However, it can be shown that this is never the case. This impossibility result is described in Lemma 1 and the proof is included in the appendix.

Lemma 1

The monopoly platform's the optimal (profit-maximizing) market shares cannot be strictly between 0 and 1 for both sides. In other words, it is impossible that $0 < n_1^* < 1, 0 < n_2^* < 1$.

In contrast to the platform literature where the optimal result is usually an interior solution, we find this is never the case under our setup. This finding provides us with a better understanding of the platform monopoly outcome with heterogeneous consumers and limited market size. Based on Lemma 1, the optimal market shares can only be in one of the following 4 situations: (1) $n_1^* = 0, n_2^* = 0$; (2) $n_1^* = 1, n_2^* = 1$; (3) $n_1^* = 1, n_2^* \in (0,1)$; (4) $n_1^* \in (0,1), n_2^* = 1$.

We define 4 exclusive market outcomes as follows, and fully characterize the monopoly's optimal solution in Proposition 1.

Outcome M1: $n_1^* = 0, n_2^* = 0$, $p_1^* \geq 0$, $p_2^* \geq 0$, $\pi^{M_1} = 0$;

Outcome M2: $n_1^* = 1, n_2^* = 1$, $p_1^* = \alpha_1 - \frac{t_1}{2}$, $p_2^* = \alpha_2 - \frac{t_2}{2}$, $\pi^{M_2} = \alpha_1 - \frac{t_1}{2} - f_1 + \alpha_2 - \frac{t_2}{2} - f_2$;

Outcome M3:

$$n_1^* = 1, n_2^* = \frac{1}{t_2}(\alpha_1 + \alpha_2 - f_2) \in (0,1) \text{ , } p_1^* = \frac{1}{t_2}(\alpha_1^2 + \alpha_1\alpha_2 - \alpha_1 f_2 - \frac{1}{2}t_1 t_2) \text{ , } p_2^* = \frac{1}{2}(\alpha_2 - \alpha_1 + f_2) \text{ ,}$$

$$\pi^{M_3} = \frac{1}{2t_2}[(\alpha_1 + \alpha_2 - f_2)^2 - (2f_1 + t_1)t_2] \text{ ;}$$

Outcome M4:

$$n_1^* = \frac{1}{t_1}(\alpha_1 + \alpha_2 - f_1) \in (0,1), n_2^* = 1 \text{ , } p_1^* = \frac{1}{2}(\alpha_1 - \alpha_2 + f_1) \text{ , } p_2^* = \frac{1}{t_1}(\alpha_2^2 + \alpha_1\alpha_2 - \alpha_2 f_1 - \frac{1}{2}t_1 t_2) \text{ ,}$$

$$\pi^{M_4} = \frac{1}{2t_1}[(\alpha_1 + \alpha_2 - f_1)^2 - (2f_2 + t_2)t_1] \text{ .}$$

Proposition 1

The monopoly platform's the optimal (profit-maximizing) market outcome can only be in one of the 4 outcomes defined above. To be more specific,

When $t_2 \geq 2f_1 + t_1$*:*

If $\alpha_1 + \alpha_2 < f_2 + \sqrt{(2f_1 + t_1)t_2}$ *, the optimal solution is characterized by Outcome M1;*

If $f_2+\sqrt{(2f_1+t_1)t_2} \le \alpha_1+\alpha_2 < t_2+f_2$, *the optimal solution is characterized by Outcome M3;*

If $\alpha_1+\alpha_2 \ge t_2+f_2$, *the optimal solution is characterized by Outcome M2.*

When $-2f_2+t_1 \le t_2 < 2f_1+t_1$*:*

If $\alpha_1+\alpha_2 < \frac{t_1+t_2}{2}+f_1+f_2$, *the optimal solution is characterized by Outcome M1;*

If $\alpha_1+\alpha_2 \ge \frac{t_1+t_2}{2}+f_1+f_2$, *the optimal solution is characterized by Outcome M2.*

When $t_2 < -2f_2+t_1$*:*

If $\alpha_1+\alpha_2 < f_1+\sqrt{(2f_2+t_2)t_1}$, *the optimal solution is characterized by Outcome M1;*

If $f_1+\sqrt{(2f_2+t_2)t_1} \le \alpha_1+\alpha_2 < t_1+f_1$, *the optimal solution is characterized by Outcome M4;*

If $\alpha_1+\alpha_2 \ge t_1+f_1$, *the optimal solution is characterized by Outcome M2.*

Proposition 1 precisely describes how cost structures and network externalities of the two-sided market jointly determine the optimal outcome. When cost structures of the two sides are relatively similar ($-f_2 \le \frac{t_2-t_1}{2} < f_1$), depending on the size of network externalities, the optimal outcome is either no coverage of service or full coverage of service, with partial coverage of service not being possible. When cost structures of the two sides are dissimilar and network externalities are of medium size, the optimal outcome has partial coverage of service for the relatively cost-inefficient side and full coverage of service for the relatively cost-efficient side.

One may be interested in the special case where the two sides of the market is symmetric. An immediate derivation from Proposition 1 provides the following result:

Corollary 1

Suppose $t_1 = t_2 = t,\ f_1 = f_2 = f,\alpha_1 = \alpha_2 = \alpha$. *The monopoly platform's the optimal (profit-maximizing) market outcome is either M1 or M2. To be more specific,*

If $\alpha < \frac{t}{2} + f$ *, the optimal solution is characterized by Outcome M1;*

If $\alpha > \frac{t}{2} + f$ *, the optimal solution is characterized by Outcome M2.*

4. Social Optimum

We now solve for the socially optimal market outcome. Similarly, $x_0 = \frac{1}{2}$ is also an optimal location for the social planer's perspective. Letting $x_0 = \frac{1}{2}$, the social planner's objective function, defined by (4), can be now represented as follows:

$$
\begin{aligned}
W^S &= (\alpha_1+\alpha_2)n_1^* n_2^* - n_1^* f_1 - n_2^* f_2 - 2\int_0^{\frac{n_1^*}{2}} n_1 t_1 dn_1 - 2\int_0^{\frac{n_2^*}{2}} n_2 t_2 dn_2 \\
&= (\alpha_1+\alpha_2)n_1^* n_2^* - n_1^* f_1 - n_2^* f_2 - (\frac{n_1^*}{2})^2 t_1 - (\frac{n_2^*}{2})^2 t_2 \qquad (7) \\
&= \underbrace{(\alpha_1+\alpha_2)n_1^* n_2^*}_{platform's\ network\ externality} - \underbrace{(n_1^* f_1 + n_2^* f_2)}_{platform's\ cost} - \underbrace{\frac{1}{4}(n_1^{*2} t_1 + n_2^{*2} t_2)}_{transport\ cost}.
\end{aligned}
$$

The social planner chooses to optimal market share profile (n_1^*, n_2^*) in order to maximize the social welfare defined by (7). A similar analysis can show that the socially optimal market share profile can never be an interior solution, described in Lemma.

Lemma 2

The socially optimal (welfare-maximizing) market shares cannot be strictly between 0 and 1 for both sides. In other words, it is impossible that $0 < n_1^* < 1, 0 < n_2^* < 1$.

Similar to the Monopoly case, since the socially optimal outcome can only be an interior solution, we define 4 exclusive market outcomes as follows, and characterize the socially optimal market shares and welfare in Proposition 2.

Outcome S1: $n_1^* = 0, n_2^* = 0, W^{S_1*} = 0$;

Outcome S2: $n_1^* = 1, n_2^* = 1, W^{S_2*} = \alpha_1 + \alpha_2 - (f_1 + f_2) - \frac{1}{4}(t_1 + t_2)$;[3]

Outcome S3: $n_1^* = 1, n_2^* = \frac{2(\alpha_1 + \alpha_2 - f_2)}{t_2} \in (0,1), W^{S_3*} = \frac{1}{4t_2}[4(\alpha_1 + \alpha_2 - f_2)^2 - (4f_1 + t_1)t_2]$;

Outcome S4: $n_1^* = \frac{2(\alpha_1 + \alpha_2 - f_1)}{t_1} \in (0,1)$, $n_2^* = 1, W^{S_4*} = \frac{1}{4t_1}[4(\alpha_1 + \alpha_2 - f_1)^2 - (4f_2 + t_2)t_1]$.

Proposition 2

The planner's socially optimal (welfare-maximizing) outcome could only be in one of the 4 outcomes defined above. To be more specific,

When $t_2 \geq 4f_1 + t_1$*:*

If $\alpha_1 + \alpha_2 < f_2 + \frac{1}{2}\sqrt{(4f_1 + t_1)t_2}$ *, the optimal solution is characterized by Outcome S1;*

If $f_2 + \frac{1}{2}\sqrt{(4f_1 + t_1)t_2} \leq \alpha_1 + \alpha_2 < \frac{1}{2}t_2 + f_2$*, the optimal solution is characterized by Outcome S3;*

If $\alpha_1 + \alpha_2 \geq \frac{1}{2}t_2 + f_2$*, the optimal solution is characterized by Outcome S2.*

When $-4f_2 + t_1 \leq t_2 < 4f_1 + t_1$*:*

If $\alpha_1 + \alpha_2 < \frac{t_1 + t_2}{4} + f_1 + f_2$*, the optimal solution is characterized by Outcome S1;*

If $\alpha_1 + \alpha_2 \geq \frac{t_1 + t_2}{4} + f_1 + f_2$*, the optimal solution is characterized by Outcome S2.*

When $t_2 < -4f_2 + t_1$*:*

[3] Note that Outcomes S1 and S2 in this section are the same as Outcomes M1 and M2 in the previous section.

If $\alpha_1+\alpha_2 < f_1+\frac{1}{2}\sqrt{(4f_2+t_2)t_1}$ *, the optimal solution is characterized by Outcome S1;*

If $f_1+\frac{1}{2}\sqrt{(4f_2+t_2)t_1} \le \alpha_1+\alpha_2 < \frac{1}{2}t_1+f_1$*, the optimal solution is characterized by Outcome S4;*

If $\alpha_1+\alpha_2 \ge \frac{1}{2}t_1+f_1$*, the optimal solution is characterized by Outcome S2.*

Similarly to Proposition 1 (monopoly case), Proposition 2 precisely describes how cost structures and network externalities of the two-sided market jointly determine the socially optimal outcome. When cost structures of the two sides are relatively similar ($-f_2 \le \frac{t_2-t_1}{4} < f_1$)[4], depending on the size of network externalities, the socially optimal outcome is either no coverage of service or full coverage of service, with partial coverage of service again not being possible. When cost structures of the two sides are dissimilar and network externalities are of medium size, the socially optimal outcome has partial coverage of service for the relatively cost-inefficient side and full coverage of service for the relatively cost-efficient side.

For the symmetric case, an immediate derivation from Proposition 2 provides the following result:

Corollary 2

Suppose $t_1=t_2=t,\ f_1=f_2=f,\alpha_1=\alpha_2=\alpha$ *. The planner's socially optimal (welfare-maximizing) market outcome is either S1 or S2. To be more specific,*

If $\alpha < \frac{t}{4}+f$ *, the optimal solution is characterized by Outcome M1;*

[4] Note that the cutoff value for similar cost structures is different between the monopoly case ($-f_2 \le \frac{t_2-t_1}{4} < f_1$) and the social optimum case ($-f_2 \le \frac{t_2-t_1}{4} < f_1$).

If $\alpha > \frac{t}{4}+f$ *, the optimal solution is characterized by Outcome M2.*

5. Welfare Comparison

Based on the results from the previous two sections, we are ready to investigate under what conditions the monopoly platform can implement a socially optimal outcome.

First, notice that social welfare only concerns about n_1^* and n_2^* without respect to p_1^* and p_2^*. Thus, for comparison between monopoly and social optimum, it suffices to focus on the market shares.

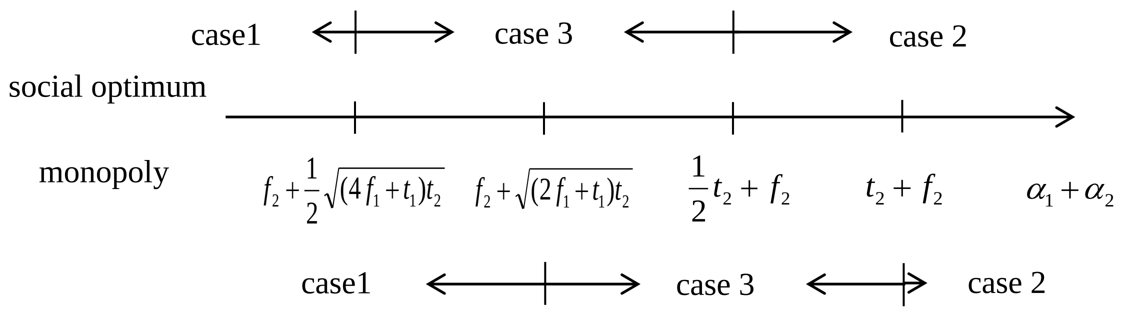
Second, it is worth noting that for Case 3 ($n_1^* = 1, n_2^* \in (0,1)$) and Case 4 ($n_1^* \in (0,1), n_2^* = 1$), the socially optimal market outcome is different from the monopoly's optimal market outcome. To be more specific, in Case 3, the socially optimal market outcome is $n_1^* = 1, n_2^* = \frac{2}{t_2}(\alpha_1 + \alpha_2 - f_2)$ while the monopoly's optimal market outcome is $n_1^* = 1, n_2^* = \frac{1}{t_2}(\alpha_1 + \alpha_2 - f_2)$. This means that in Case 3, the monopoly's market share for group 2 is equal to half size of the socially optimal level. A similar result holds for Case 4, where the socially optimal market outcome is $n_1^* = \frac{2}{t_1}(\alpha_1 + \alpha_2 - f_1), n_2^* = 1$ while the monopoly's optimal market outcome is $n_1^* = \frac{1}{t_1}(\alpha_1 + \alpha_2 - f_1), n_2^* = 1$. Thus, to see if the monopoly platform can implement a socially optimal outcome, it suffices to look for the conditions under which the optimal market outcomes for the monopoly and the social planner are either both Case 1 or both Case 2. Since Case 1 simply represents the no coverage of service outcome, which is not of our interest, the conditions that guarantee Case 2 occurs are the conditions we are essentially searching for.

By comparing the results in Propositions 1 and 2, we have in total 9 scenarios to consider.[5] In the following, we conduct the comparison analysis for each scenario.

CI) $t_2 > 4(2f_1 + t_1)$:

Note that $t_2 > 4(2f_1 + t_1)$ implies $\frac{1}{2}t_2 + f_2 > f_2 + \sqrt{(2f_1 + t_1)t_2}$ and $f_2 + \sqrt{(2f_1 + t_1)t_2} > f_2 + \frac{1}{2}\sqrt{(4f_1 + t_1)t_2}$ always holds. For all possible values of $\alpha_1 + \alpha_2$, we compare the optimal market outcomes for social planner and for monopolist. The results are summarized in the following figure:

Figure 2 **Welfare Comparison for** $t_2 > 4(2f_1 + t_1)$



Based on Figure 2, we have the following observations:

① When $\alpha_1 + \alpha_2 \in (0, f_2 + \frac{1}{2}\sqrt{t_2(4f_1 + t_1)})$, externalities are so small that neither the monopolist nor the social planer provides any service (the M1/S1 outcome).

② When $\alpha_1 + \alpha_2 \in (f_2 + \frac{1}{2}\sqrt{t_2(4f_1 + t_1)},\ f_2 + \sqrt{t_2(2f_1 + t_1)})$, the monopolist does not want to enter the market because externalities are not sufficiently large to offset the costs (the M1 outcome), while the externalities are large enough for the social planner to provide full service to group 1 and partial service to group 2 (the S3 outcome).

[5] For simplicity, we did not consider the scenarios where cutoff conditions hold with equality.

③ When $\alpha_1+\alpha_2 \in (f_2+\sqrt{t_2(2f_1+t_1)}, \frac{1}{2}t_2+f_2)$, both the monopolist and the social planner provide full service to group 1 and partial service to group 2 (the M3/S3 outcome). However, the partial coverage of group 2 agents is different between the monopolist and the social planer.

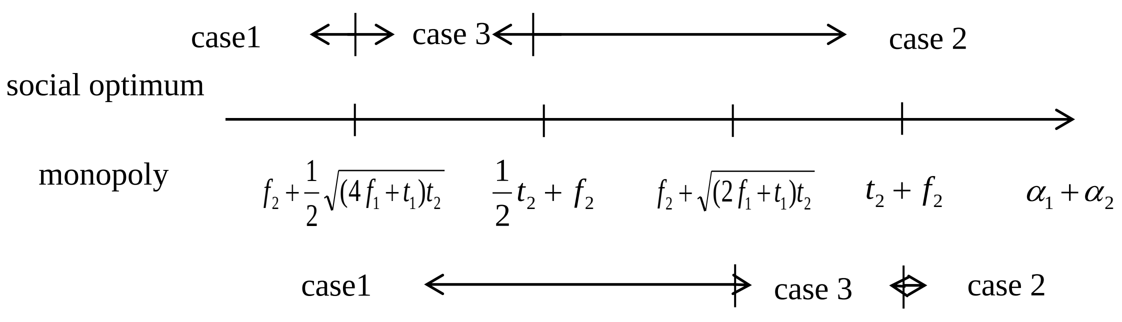
④ When $\alpha_1+\alpha_2 \in (\frac{1}{2}t_2+f_2, t_2+f_2)$, the monopolist provides full service to group 1 and partial service to group 2 (the M3 outcome), while the social planner provides full service to both sides of the market (the S2 outcome).

⑤ When $\alpha_1+\alpha_2 \in (t_2+f_2, +\infty)$, the externalities are so large that both the monopolist and the social planner choose to provide full service to both sides of the market. (the M2/S2 outcome). This is the case where the monopoly is socially optimal.

CII) $4f_1+t_1<t_2<4(2f_1+t_1)$:

Note that $t_2<4(2f_1+t_1)$ implies $\frac{1}{2}t_2+f_2 < f_2+\sqrt{(2f_1+t_1)t_2}$. We conduct the similar comparison and obtain the following figure:

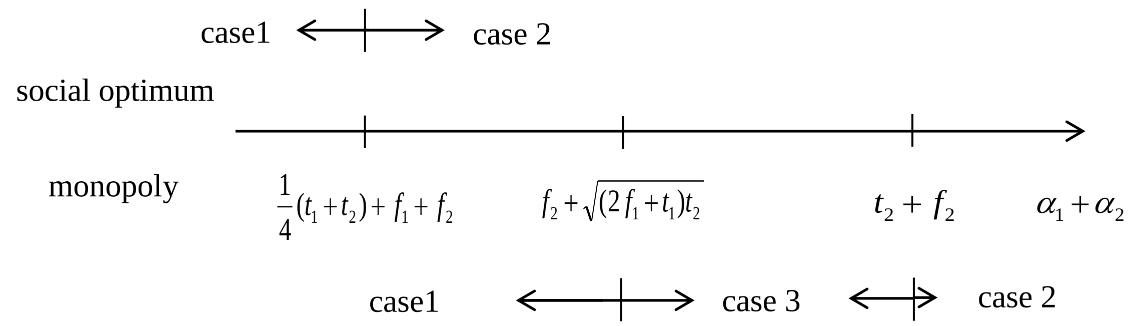
Figure 3 Welfare Comparison for $4f_1+t_1<t_2<4(2f_1+t_1)$



CIII) $\max\{2f_1+t_1, 12f_1+7t_1-4\sqrt{(2f_1+t_1)(4f_1+3t_1)}\}<t_2<4f_1+t_1$:

Note that $12f_1 + 7t_1 + 4\sqrt{(2f_1 + t_1)(4f_1 + 3t_1)} > t_2 > 12f_1 + 7t_1 - 4\sqrt{(2f_1 + t_1)(4f_1 + 3t_1)}$ implies $\frac{1}{4}(t_1 + t_2) + f_1 + f_2 < f_2 + \sqrt{(2f_1 + t_1)t_2}$. Also note that $12f_1 + 7t_1 + 4\sqrt{(2f_1 + t_1)(4f_1 + 3t_1)} > 4f_1 + t_1$ always holds. We conduct the similar comparison and obtain the following figure:

Figure 4 Welfare Comparison for $\max\{2f_1 + t_1, 12f_1 + 7t_1 - 4\sqrt{(2f_1 + t_1)(4f_1 + 3t_1)}\} < t_2 < 4f_1 + t_1$

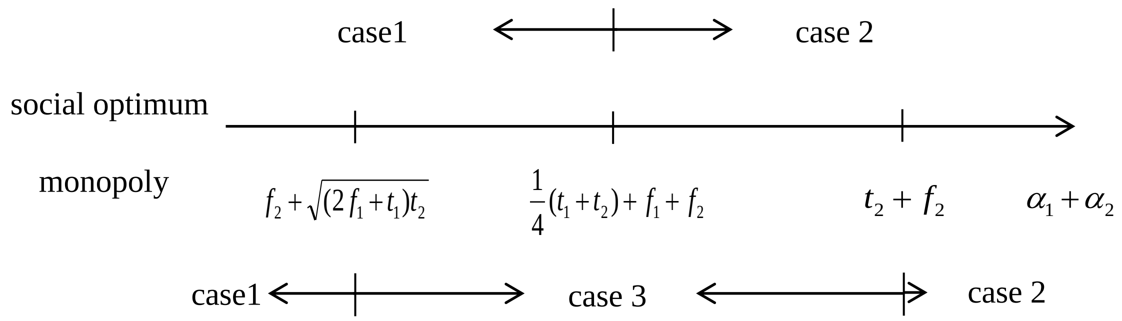


CIV) $2f_1 + t_1 < t_2 < \min\{4f_1 + t_1, 12f_1 + 7t_1 - 4\sqrt{(2f_1 + t_1)(4f_1 + 3t_1)}\}$:

Note that $t_2 < 12f_1 + 7t_1 - 4\sqrt{(2f_1 + t_1)(4f_1 + 3t_1)}$ implies $\frac{1}{4}(t_1 + t_2) + f_1 + f_2 > f_2 + \sqrt{(2f_1 + t_1)t_2}$.

Also note that $2f_1 + t_1 < t_2$ implies $\frac{1}{4}(t_1 + t_2) + f_1 + f_2 < t_2 + f_2$. We conduct the similar comparison and obtain the following figure:

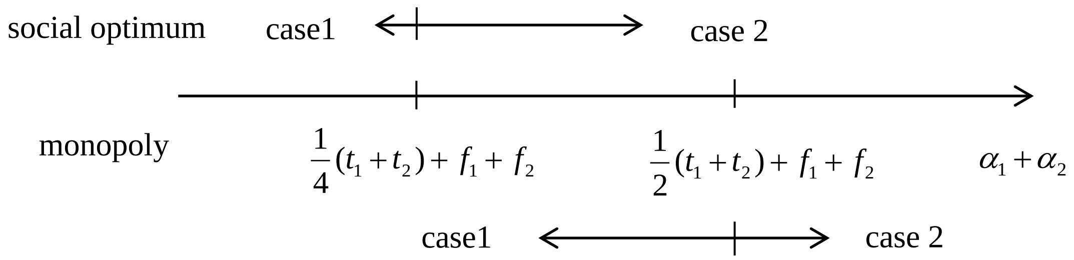
Figure 5 Welfare Comparison for $2f_1 + t_1 < t_2 < \min\{4f_1 + t_1, 12f_1 + 7t_1 - 4\sqrt{(2f_1 + t_1)(4f_1 + 3t_1)}\}$



CV) $-2f_2 + t_1 < t_2 < 2f_1 + t_1$:

Note that $\frac{t_1+t_2}{4}+f_1+f_2<\frac{t_1+t_2}{2}+f_1+f_2$ always holds. We conduct the similar comparison and obtain the following figure:

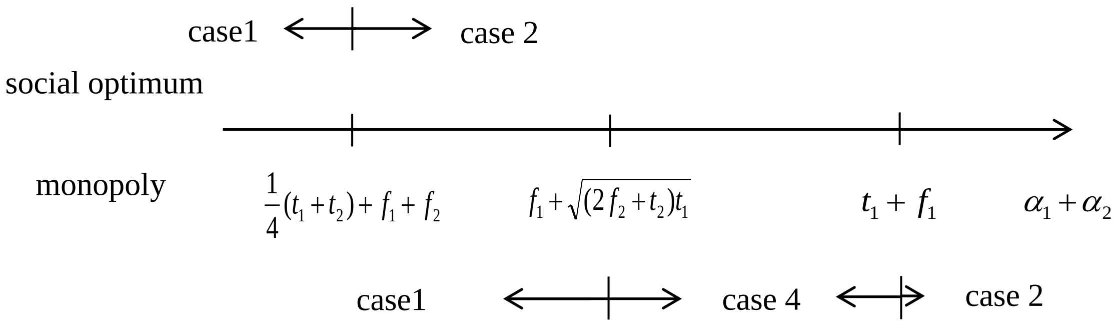
Figure 6 **Welfare Comparison for** $-2f_2+t_1<t_2<2f_1+t_1$



CVI) $\max\{-4f_2+t_1,-4f_2+7t_1-4\sqrt{t_1(-2f_2+3t_1)}\}<t_2<-2f_2+t_1$:

Note that $-4f_2+7t_1+4\sqrt{t_1(-2f_2+3t_1)}>t_2>-4f_2+7t_1-4\sqrt{t_1(-2f_2+3t_1)}$ implies $\frac{1}{4}(t_1+t_2)+f_1+f_2<f_1+\sqrt{(2f_2+t_2)t_1}$. Also note that $-4f_2+7t_1+4\sqrt{t_1(-2f_2+3t_1)}>-2f_2+t_1$ always holds. We conduct the similar comparison and obtain the following figure:

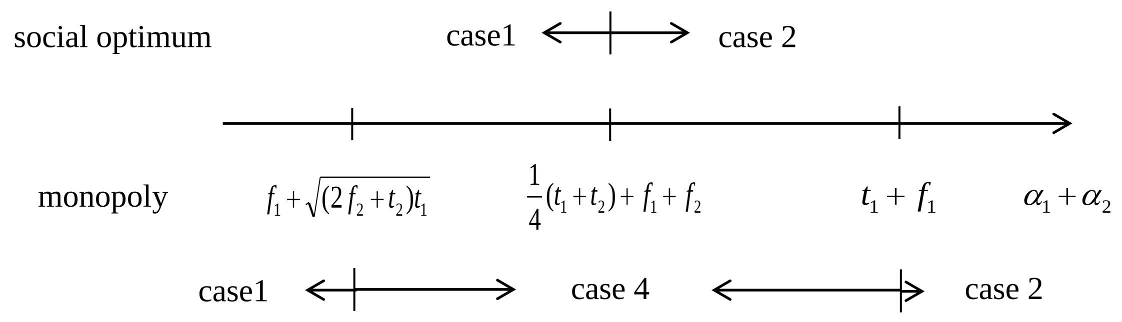
Figure 7 **Welfare Comparison for** $\max\{-4f_2+t_1,-4f_2+7t_1-4\sqrt{t_1(-2f_2+3t_1)}\}<t_2<-2f_2+t_1$



CVII) $-4f_2+t_1<t_2<\min\{-2f_2+t_1,-4f_2+7t_1-4\sqrt{t_1(-2f_2+3t_1)}\}$:

Note that $t_2<-4f_2+7t_1-4\sqrt{t_1(-2f_2+3t_1)}$ implies $\frac{1}{4}(t_1+t_2)+f_1+f_2>f_1+\sqrt{(2f_2+t_2)t_1}$. Also note that $t_2 < -2f_2 + t_1$ implies $\frac{1}{4}(t_1 + t_2) + f_1 + f_2 < t_1 + f_1$. We conduct the similar comparison and obtain the following figure:

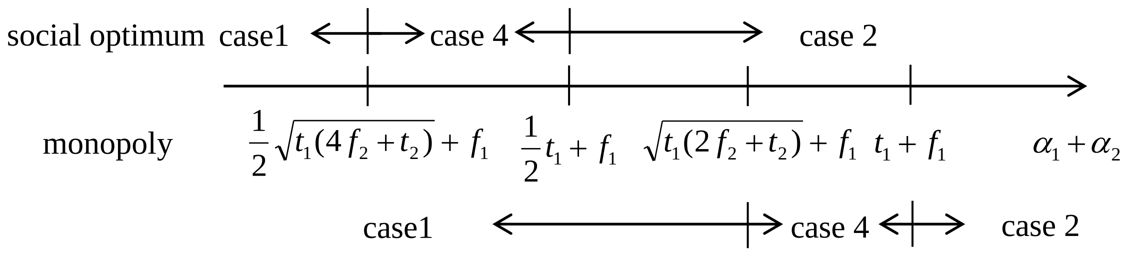
Figure 8 **Welfare Comparison for** $-4f_2 + t_1 < t_2 < \min\{-2f_2 + t_1, -4f_2 + 7t_1 - 4\sqrt{t_1(-2f_2 + 3t_1)}\}$



CVIII) $\frac{t_1}{4} - 2f_2 < t_2 < -4f_2 + t_1$:

Note that $t_2 > \frac{t_1}{4} - 2f_2$ implies $\frac{1}{2}t_1 + f_1 < f_1 + \sqrt{(2f_2 + t_2)t_1}$. We conduct the similar comparison and obtain the following figure:

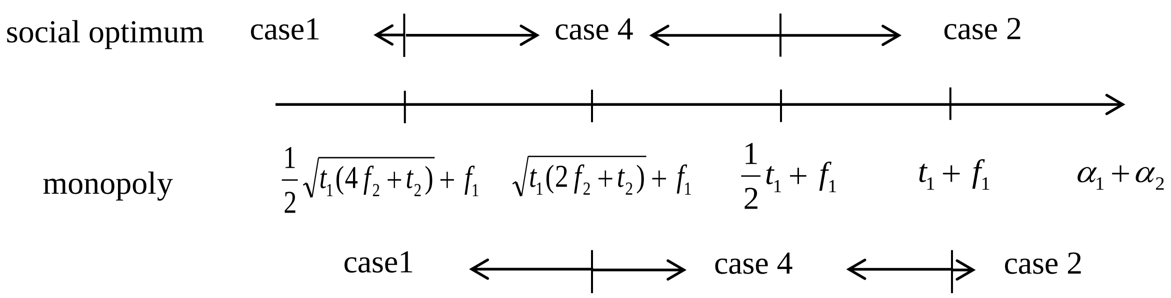
Figure 9 **Welfare Comparison for** $\frac{t_1}{4} - 2f_2 < t_2 < -4f_2 + t_1$



CIX) $t_2 < \frac{t_1}{4} - 2f_2$:

Note that $t_2 < \frac{t_1}{4} - 2f_2$ implies $\frac{1}{2}t_1 + f_1 > f_1 + \sqrt{(2f_2 + t_2)t_1}$ and $f_1 + \sqrt{(2f_2 + t_2)t_1} > f_1 + \frac{1}{2}\sqrt{(4f_2 + t_2)t_1}$ always holds. We conduct the similar comparison and obtain the following figure:

Figure 10 Welfare Comparison for $t_2 < \frac{t_1}{4} - 2f_2$



From all the comparison results above, we have the following observations:

(1) When cost structures of the two sides are dissimilar: The cutoff value of $\alpha_1+\alpha_2$ between Case 1 and Case 3/4 is greater for monopoly than for social optimum, implying a higher level of network externality is needed for the monopolist than for the social planner to switch from providing no service to partially serving the market; The cutoff value of $\alpha_1+\alpha_2$ between Case 3/4 and Case 2 is greater for monopoly than for social optimum, implying a higher level of network externality is needed for the monopolist than for the social planner to switch from partially serving the market to fully serving the market on both sides.

(2) When cost structures of the two sides are similar: The cutoff value of $\alpha_1+\alpha_2$ between Case 1 and Case 2 is greater for monopoly than for social optimum, implying a higher level of network externality is needed for the monopolist than for the social planner to switch from providing no service to fully serving the market on both sides.

(3) When $\alpha_1+\alpha_2 < \min\{f_1+\frac{1}{2}\sqrt{(4f_2+t_2)t_1},\, f_2+\frac{1}{2}\sqrt{(4f_1+t_1)t_2},\, f_1+\sqrt{(2f_2+t_2)t_1},\, f_2+\sqrt{(2f_1+t_1)t_2},\, \frac{1}{4}(t_1+t_2)+f_1+f_2\}$, the externality level is so low that neither the monopolist nor the social planner provides any service. In this case monopoly is socially optimal.

(4) When $\alpha_1+\alpha_2 > \max\{t_1+f_1, t_2+f_2, \frac{1}{2}(t_1+t_2)+f_1+f_2\}$, the externality level is so high that both the monopolist and the social planner fully serve the market on both sides. In this case monopoly is socially optimal.

(5) When $\min\{f_1+\frac{1}{2}\sqrt{(4f_2+t_2)t_1}, f_2+\frac{1}{2}\sqrt{(4f_1+t_1)t_2}, f_1+\sqrt{(2f_2+t_2)t_1}, f_2+\sqrt{(2f_1+t_1)t_2}, \frac{1}{4}(t_1+t_2)+f_1+f_2\} < \alpha_1+\alpha_2 < \max\{t_1+f_1, t_2+f_2, \frac{1}{2}(t_1+t_2)+f_1+f_2\}$, the externality level is of medium size, and the monopoly market share and the socially optimal market share differ. In this case, the monopolist would distort the resource allocations to maximize its profit, resulting in a social welfare loss.

We now summarize the conditions under which monopoly is socially optimal, in the following proposition.

Proposition 3 (Socially Optimal Monopoly)

The monopoly platform is socially optimal only if $n_1^*=0, n_2^*=0$ *or* $n_1^*=1, n_2^*=1$. *Focusing on the* $n_1^*=1, n_2^*=1$ *case, to be more specific, both the monopolist and the social planner provide full service to both sides of the market, if and only if one of the following 3 conditions holds: (1)* $t_2 > 2f_1+t_1$ and $\alpha_1+\alpha_2 > t_2+f_2$; (2) $-2f_2+t_1 < t_2 < 2f_1+t_1$ and $\alpha_1+\alpha_2 > \frac{1}{2}(t_1+t_2)+f_1+f_2$; (3) $t_2 < -2f_2+t_1$ and $\alpha_1+\alpha_2 > t_1+f_1$.

For the symmetric case, an immediate derivation from Proposition 3 and the above analysis provides the following result:

Corollary 3

Suppose $t_1=t_2=t, f_1=f_2=f, \alpha_1=\alpha_2=\alpha$. *When* $\alpha < \frac{t}{4}+f$ *, neither the monopolist nor the* *social planner provides any service; When* $\frac{t}{4}+f<\alpha<\frac{t}{2}+f$ *, the monopolist provides no service while the social planner fully serves the market; When* $\alpha>\frac{t}{2}+f$ *, both the monopolist and the social planner fully serve the market.*

6. Conclusion

We study a monopoly platform's optimal pricing strategy in an environment of limited market size, heterogeneous agents, and endogenous demand, with the objective to maximize platform profit or social welfare, respectively. With each objective, we fully characterize the optimal strategy of the monopoly platform, which depends both on the cost structure of the two-side market and the degree of the network externalities. When cost structures of the two sides are relatively similar, the optimal outcome is no coverage of service if the externalities are small and full coverage of service if the externalities are large. When cost structures of the two sides are dissimilar, and network externalities are of medium size, the optimal outcome has partial coverage of service for the relatively cost-inefficient side and full coverage of service for the relatively cost-efficient side. Comparing the results for the monopoly case and the social optimum case, we provide necessary and sufficient conditions under which the monopoly outcome is socially optimal.

For future work, one potential direction will be to consider the case where the agents' network utility does not only depend the size of the other side but also depend on the size of the same size, which can better capture the reality of competition within the same group. Another potential direction is to extend the monopoly model to the duopoly model with competition between platforms where the market size is not fixed.

Appendix

I. Proof of Lemma 1

We want to show that the monopoly's optimal (profit-maximizing) market shares cannot be strictly between 0 and 1 for both sides. In other words, it is impossible that $0 < n_1^* < 1, 0 < n_2^* < 1$.

Suppose not. Then by demand functions $n_1 = \frac{2}{t_1}(\alpha_1 n_2 - p_1)$ and $n_2 = \frac{2}{t_2}(\alpha_2 n_1 - p_2)$, we can obtain the following two equations:

$$n_1 = \frac{2(2\alpha_1 p_2 + t_2 p_1)}{4\alpha_1\alpha_2 - t_1 t_2}, \tag{A1}$$

$$n_2 = \frac{2(2\alpha_2 p_1 + t_1 p_2)}{4\alpha_1\alpha_2 - t_1 t_2}. \tag{A2}$$

Then, we can write the profit of the platform as a function of p_1 and p_2:

$$\pi^M = \left(\frac{2(2\alpha_1 p_2 + t_2 p_1)}{4\alpha_1\alpha_2 - t_1 t_2}\right)(p_1 - f_1) + \left(\frac{2(2\alpha_2 p_1 + t_1 p_2)}{4\alpha_1\alpha_2 - t_1 t_2}\right)(p_2 - f_2). \tag{A3}$$

Then, we use the first order conditions to derive the optimal prices equations for the platform:

$$p_1 = \frac{1}{2}\left(f_1 - \frac{2(\alpha_1 + \alpha_2)p_2 - 2\alpha_2 f_2}{t_2}\right), \tag{A4}$$

$$p_2 = \frac{1}{2}\left(f_2 - \frac{2(\alpha_1 + \alpha_2)p_1 - 2\alpha_1 f_1}{t_1}\right). \tag{A5}$$

Solving the above two equations simultaneously, we obtain the optimal prices:

$$p_1^* = \frac{1}{2}\frac{2\alpha_1 f_1(\alpha_1 + \alpha_2) + \alpha_1 f_2 t_1 - \alpha_2 f_2 t_1 - f_1 t_1 t_2}{(\alpha_1 + \alpha_2)^2 - t_1 t_2}, \tag{A6}$$

$$p_2^* = \frac{1}{2}\frac{2\alpha_2 f_2(\alpha_1 + \alpha_2) + \alpha_2 f_1 t_2 - \alpha_1 f_1 t_2 - f_2 t_1 t_2}{(\alpha_1 + \alpha_2)^2 - t_1 t_2}. \tag{A7}$$

Expressing n_1, n_2 in terms of p_1^*, p_2^*, (A1) and (A2) become:

$$n_1^* = \frac{f_2(\alpha_1+\alpha_2)+f_1t_2}{(\alpha_1+\alpha_2)^2-t_1t_2}, \tag{A8}$$

$$n_2^* = \frac{f_1(\alpha_1+\alpha_2)+f_2t_1}{(\alpha_1+\alpha_2)^2-t_1t_2}. \tag{A9}$$

Then, we plug $n_1^*, n_2^*, p_1^*, p_2^*$ into π^M and obtain the following optimal profit:

$$\pi^{M*} = -\frac{1}{2}\frac{2\alpha_1 f_1 f_2 + 2\alpha_2 f_1 f_2 + f_1^2 t_2 + f_2^2 t_1}{(\alpha_1+\alpha_2)^2-t_1t_2}. \tag{A10}$$

Since $n_1^*, n_2^* \in (0,1)$ requires $(\alpha_1+\alpha_2)^2 - t_1t_2 > 0$ by (A8) or (A9), we have $\pi^{M*} < 0$. Thus, the above result is not optimal for the monopoly platform, since it is worse than $\pi^M = 0$ where the platform does not attract any agents on the two-sided market.

QED.

II. Proof of Proposition 1

Step 1: We show that the optimal market outcome can only be in one of the following 4 cases: Case 1: $n_1^* = 0, n_2^* = 0$; Case 2: $n_1^* = 1, n_2^* = 1$; Case 3: $n_1^* = 1, n_2^* \in (0,1)$; and Case 4: $n_1^* \in (0,1), n_2^* = 1$.

It suffices to show if $n_1^* n_2^* = 0$, then $n_1^* = n_2^* = 0$. Without loss of generality, suppose $n_1^* = 0$. Since the platform's profit is $n_1(p_1 - f_1) + n_2(p_2 - f_2)$, when $n_1^* = 0$, the profit reduces to $n_2(p_2 - f_2)$.

If $n_2^* = 1$, then in order to maximize the profit we must have $p_2^* \geq f_2 > 0$. However, in this case no agents in group 2 will be attracted to the platform as $u_2 = \alpha_2 n_1^* - p_2^* < 0$, which contracts with $n_2^* = 1$.

If $n_2^* \in (0,1)$, then by the demand function $n_2^* = \frac{2}{t_2}(\alpha_2 n_1^* - p_2^*) = -\frac{2}{t_2} p_2^*$, we know $p_2^* < 0$.

Thus, the optimal profit $-\frac{2}{t_2}p_2^*(p_2^*-f_2)$ is negative, which again leads to a contradiction.

Therefore, $n_1^*=0$ implies $n_2^*=0$, and vice versa.

Step 2: We completely characterize the optimal outcome for the monopoly platform in each case.

Case 1: $n_1^*=0, n_2^*=0$

$n_1^*=0, n_2^*=0$ implies $p_1' \geq \alpha_1 n_2^*$ and $p_2' \geq \alpha_2 n_1^*$. For the optimal prices we have $p_1^* \geq 0$ and $p_2^* \geq 0$. The optimal profit is $\pi^{M_1}=0$.

Case 2: $n_1^*=1, n_2^*=1$

$n_1^*=1, n_2^*=1$ implies $p_1' \leq \alpha_1 n_2^* - \frac{t_1}{2} = \alpha_1 - \frac{t_1}{2}$ and $p_2' \leq \alpha_2 n_1^* - \frac{t_2}{2} = \alpha_2 - \frac{t_2}{2}$. For the optimal prices we should have $p_1^* = \alpha_1 - \frac{t_1}{2}$ and $p_2^* = \alpha_2 - \frac{t_2}{2}$. The optimal profit is $\pi^{M_2} = \alpha_1 - \frac{t_1}{2} - f_1 + \alpha_2 - \frac{t_2}{2} - f_2$. Note that $\pi^{M_2} \geq 0$ requires $\alpha_1 + \alpha_2 \geq \frac{t_1+t_2}{2} + f_1 + f_2$. Also, from the analyses below in Case 3 and Case 4, we will see that $n_1^*=1$ requires $\alpha_1 + \alpha_2 \geq t_1 + f_1$ and $n_2^*=1$ requires $\alpha_1 + \alpha_2 \geq t_2 + f_2$.

Case 3: $n_1^*=1, n_2^* \in (0,1)$

$n_1^*=1, n_2^* \in (0,1)$ implies $p_1' \leq \alpha_1 n_2^* - \frac{t_1}{2}$ and $\alpha_2 n_1^* - \frac{t_2}{2} = \alpha_2 - \frac{t_2}{2} < p_2' < \alpha_2 n_1^* = \alpha_2$. For the optimal prices we should have $p_1^* = \alpha_1 n_2^* - \frac{t_1}{2}$ and $\alpha_2 - \frac{t_2}{2} < p_2^* < \alpha_2$ such that $0 < n_2^* = \frac{2}{t_2}(\alpha_2 - p_2^*) < 1$. Thus, the optimal profit is $\pi^{M_3} = p_1^* - f_1 + \frac{2}{t_2}(\alpha_2 - p_2^*)(p_2^* - f_2)$, further expressed as a function of p_2^*: $\pi^{M_3} = \alpha_1 \frac{2}{t_2}(\alpha_2 - p_2^*) - \frac{t_1}{2} - f_1 + \frac{2}{t_2}(\alpha_2 - p_2^*)(p_2^* - f_2)$.

By the first order condition, we have $p_2^* = \frac{1}{2}(\alpha_2 - \alpha_1 + f_2)$. Thus, $n_2^* = \frac{1}{t_2}(\alpha_1 + \alpha_2 - f_2)$, $p_1^* = \frac{1}{t_2}(\alpha_1^2 + \alpha_1\alpha_2 - \alpha_1 f_2 - \frac{1}{2}t_1 t_2)$, and $\pi^{M_3} = \frac{1}{2t_2}[(\alpha_1 + \alpha_2 - f_2)^2 - (2f_1 + t_1)t_2]$. Note that $n_2^* \in (0,1)$ requires $f_2 < \alpha_1 + \alpha_2 < t_2 + f_2$. Also note that $\pi^{M_3} > 0$ requires $\alpha_1 + \alpha_2 > f_2 + \sqrt{(2f_1 + t_1)t_2}$.

Case 4: $n_2^* = 1, n_1^* \in (0,1)$

$n_2^* = 1, n_1^* \in (0,1)$ implies $p_2' \le \alpha_2 n_1^* - \frac{t_2}{2}$ and $\alpha_1 n_2^* - \frac{t_1}{2} = \alpha_1 - \frac{t_1}{2} < p_1' < \alpha_1 n_2^* = \alpha_1$. For the optimal prices we should have $p_2^* = \alpha_2 n_1^* - \frac{t_2}{2}$ and $\alpha_1 - \frac{t_1}{2} < p_1^* < \alpha_1$ such that $0 < n_1^* = \frac{2}{t_1}(\alpha_1 - p_1^*) < 1$. Thus, the optimal profit is $\pi^{M_4} = \frac{2}{t_1}(\alpha_1 - p_1^*)(p_1^* - f_1) + p_2^* - f_2$, further expressed as a function of p_1^*: $\pi^{M_4} = \frac{2}{t_1}(\alpha_1 - p_1^*)(p_1^* - f_1) + (\alpha_2 \frac{2}{t_1}(\alpha_1 - p_1^*) - \frac{t_2}{2}) - f_2$.

By the first order condition, we have $p_1^* = \frac{1}{2}(\alpha_1 - \alpha_2 + f_1)$. Thus, $n_1^* = \frac{1}{t_1}(\alpha_1 + \alpha_2 - f_1)$, $p_2^* = \frac{1}{t_1}(\alpha_2^2 + \alpha_1\alpha_2 - \alpha_2 f_1 - \frac{1}{2}t_1 t_2)$, and $\pi^{M_4} = \frac{1}{2t_1}[(\alpha_1 + \alpha_2 - f_1)^2 - (2f_2 + t_2)t_1]$. Note that $n_1^* \in (0,1)$ requires $f_1 < \alpha_1 + \alpha_2 < t_1 + f_1$. Also note that $\pi^{M_4} > 0$ requires $\alpha_1 + \alpha_2 > f_1 + \sqrt{(2f_2 + t_2)t_1}$.

Also, by simple algebraic operations, we can show $\pi^{M_3} - \pi^{M_2} = \frac{1}{2}\frac{(\alpha_1 + \alpha_2 - f_2 - t_2)^2}{t_2} \ge 0$ and $\pi^{M_4} - \pi^{M_2} = \frac{1}{2}\frac{(\alpha_1 + \alpha_2 - f_1 - t_1)^2}{t_1} \ge 0$, where the two equalities hold only when

$\alpha_1+\alpha_2=t_2+f_2$ and $\alpha_1+\alpha_2=t_1+f_1$, respectively.

Step 3: We completely characterize the optimal outcome for the monopoly platform under full parameter space.

MI) $t_2 \ge 2f_1+t_1$:

In this scenario, we have $\sqrt{(2f_1+t_1)t_2} \le t_2$ and $t_2+f_2 \ge t_1+f_1$.

Note that Case 4 is not optimal since $\alpha_1+\alpha_2<t_1+f_1$ and $\alpha_1+\alpha_2>f_1+\sqrt{(2f_2+t_2)t_1}$ imply $2f_2+t_2<t_1$, which contradicts with $t_2 \ge 2f_1+t_1$.

MI-i) $\alpha_1+\alpha_2<f_2+\sqrt{(2f_1+t_1)t_2}$:

Case 3 is not optimal since $\alpha_1+\alpha_2<f_2+\sqrt{(2f_1+t_1)t_2}$ implies $\pi^{M_3}<0$. Case 2 is not optimal since $\pi^{M_2} \le \pi^{M_3}<0$. Therefore, the optimal solution is characterized by Case 1.

MI-ii) $f_2+\sqrt{(2f_1+t_1)t_2} \le \alpha_1+\alpha_2<t_2+f_2$:

Case 2 is not optimal since $\alpha_1+\alpha_2<t_2+f_2$ implies $n_2^*<1$. Case 1 is not optimal since $\alpha_1+\alpha_2 \ge f_2+\sqrt{(2f_1+t_1)t_2}$ implies $\pi^{M_3} \ge 0$. Therefore, the optimal solution is characterized by Case 3.

MI-iii) $\alpha_1+\alpha_2 \ge t_2+f_2$:

Case 3 is not optimal since $n_2^* \in (0,1)$ requires $\alpha_1+\alpha_2<t_2+f_2$, contradicting $\alpha_1+\alpha_2 \ge t_2+f_2$. Case 1 is not optimal since $\alpha_1+\alpha_2 \ge t_2+f_2$ implies $\alpha_1+\alpha_2 \ge \frac{t_1+t_2}{2}+f_1+f_2$ given $t_2 \ge 2f_1+t_1$, and $\alpha_1+\alpha_2 \ge \frac{t_1+t_2}{2}+f_1+f_2$ implies $\pi^{M_2} \ge 0$. Therefore, the optimal solution is characterized by Case 2.

MII) $f_1-f_2+t_1 \le t_2<2f_1+t_1$:

In this scenario, we have $\sqrt{(2f_1+t_1)t_2} > t_2$ and $t_2 + f_2 \geq t_1 + f_1$.

Note that Case 4 is not optimal since $\alpha_1 + \alpha_2 < t_1 + f_1$ and $\alpha_1 + \alpha_2 > f_1 + \sqrt{(2f_2+t_2)t_1}$ imply $2f_2 + t_2 < t_1$, which contradicts with $f_1 - f_2 + t_1 \leq t_2$.

Also note that Case 3 is not optimal since $\alpha_1 + \alpha_2 < t_2 + f_2$ and $\alpha_1 + \alpha_2 > f_2 + \sqrt{(2f_1+t_1)t_2}$ imply $t_2 > 2f_1 + t_1$, which contradicts with $t_2 < 2f_1 + t_1$.

MII-i) $\alpha_1 + \alpha_2 < \frac{t_1+t_2}{2} + f_1 + f_2$:

Case 2 is not optimal since $\alpha_1 + \alpha_2 < \frac{t_1+t_2}{2} + f_1 + f_2$ implies $\pi^{M_2} < 0$. Therefore, the optimal solution is characterized by Case 1.

MII-ii) $\alpha_1 + \alpha_2 \geq \frac{t_1+t_2}{2} + f_1 + f_2$:

$\alpha_1 + \alpha_2 \geq \frac{t_1+t_2}{2} + f_1 + f_2$ and $t_2 < 2f_1 + t_1$ imply that $\alpha_1 + \alpha_2 \geq f_2 + t_2$. $\alpha_1 + \alpha_2 \geq f_2 + t_2$ and $t_2 + f_2 \geq t_1 + f_1$ imply $\alpha_1 + \alpha_2 \geq f_1 + t_1$. Thus we have $\pi^{M_2} \geq 0$ and $n_1^* = 1, n_2^* = 1$. Therefore, the optimal solution is characterized by Case 2.

Similarly, we can show the results when $f_1 + t_1 > f_2 + t_2$.

MIII) $-2f_2 + t_1 \leq t_2 < f_1 - f_2 + t_1$

In this scenario, we have $\sqrt{(2f_2+t_2)t_1} \geq t_1$ and $t_2 + f_2 < t_1 + f_1$.

Note that Case 3 is not optimal since $\alpha_1 + \alpha_2 < t_2 + f_2$ and $\alpha_1 + \alpha_2 > f_2 + \sqrt{(2f_1+t_1)t_2}$ imply $t_2 > 2f_1 + t_1$, which contradicts with $t_2 < f_1 - f_2 + t_1$.

Also note that Case 4 is not optimal since $\alpha_1 + \alpha_2 < t_1 + f_1$ and $\alpha_1 + \alpha_2 > f_1 + \sqrt{(2f_2+t_2)t_1}$ imply $2f_2 + t_2 < t_1$, which contradicts with $-2f_2 + t_1 \leq t_2$.

MIII-i) $\alpha_1+\alpha_2<\frac{t_1+t_2}{2}+f_1+f_2$:

Case 2 is not optimal since $\alpha_1+\alpha_2<\frac{t_1+t_2}{2}+f_1+f_2$ implies $\pi^{M_2}<0$. Therefore, the optimal solution is characterized by Case 1.

MIII-ii) $\alpha_1+\alpha_2\geq\frac{t_1+t_2}{2}+f_1+f_2$:

$\alpha_1+\alpha_2\geq\frac{t_1+t_2}{2}+f_1+f_2$ and $t_2\geq-2f_2+t_1$ imply that $\alpha_1+\alpha_2\geq f_1+t_1$. $\alpha_1+\alpha_2\geq f_1+t_1$ and $f_1+t_1>f_2+t_2$ imply $\alpha_1+\alpha_2>f_2+t_2$. Thus we have $\pi^{M_2}\geq 0$ and $n_1^*=1, n_2^*=1$. Therefore, the optimal solution is characterized by Case 2.

MIV) $t_2<-2f_2+t_1$

In this scenario, we have $\sqrt{(2f_2+t_2)t_1}<t_1$ and $t_2+f_2<t_1+f_1$.

Note that Case 3 is not optimal since $\alpha_1+\alpha_2<t_2+f_2$ and $\alpha_1+\alpha_2>f_2+\sqrt{(2f_1+t_1)t_2}$ imply $t_2>2f_1+t_1$, which contradicts with $t_2<-2f_2+t_1$.

MIV-i) $\alpha_1+\alpha_2<f_1+\sqrt{(2f_2+t_2)t_1}$:

Case 4 is not optimal since $\alpha_1+\alpha_2<f_1+\sqrt{(2f_2+t_2)t_1}$ implies $\pi^{M_4}<0$. Case 2 is not optimal since $\pi^{M_2}\leq\pi^{M_4}<0$. Therefore, the optimal solution is characterized by Case 1.

MIV-ii) $f_1+\sqrt{(2f_2+t_2)t_1}\leq\alpha_1+\alpha_2<t_1+f_1$:

Case 2 is not optimal since $\alpha_1+\alpha_2<t_1+f_1$ implies $n_1^*<1$. Case 1 is not optimal since $\alpha_1+\alpha_2\geq f_1+\sqrt{(2f_2+t_2)t_1}$ implies $\pi^{M_4}\geq 0$. Therefore, the optimal solution is characterized by Case 4.

MIV-iii) $\alpha_1+\alpha_2\geq t_1+f_1$:

Case 4 is not optimal since $n_1^* \in (0,1)$ requires $\alpha_1 + \alpha_2 < t_1 + f_1$, contradicting $\alpha_1 + \alpha_2 \geq t_1 + f_1$. Case 1 is not optimal since $\alpha_1 + \alpha_2 \geq t_1 + f_1$ implies $\alpha_1 + \alpha_2 > \frac{t_1+t_2}{2}+f_1+f_2$ given $t_2 < -2f_2 + t_1$, and $\alpha_1 + \alpha_2 > \frac{t_1+t_2}{2}+f_1+f_2$ implies $\pi^{M_2} > 0$. Therefore, the optimal solution is characterized by Case 2.

QED.

III. Proof of Lemma 2

We want to show that the socially optimal (welfare-maximizing) market shares cannot be strictly between 0 and 1 for both sides. In other words, it is impossible that $0 < n_1^* < 1, 0 < n_2^* < 1$.

Suppose not. Then by maximizing the welfare function $W^S = (\alpha_1 + \alpha_2)n_1 n_2 - (n_1 f_1 + n_2 f_2) - \frac{1}{4}(n_1^2 t_1 + n_2^2 t_2)$ with respect to n_1 and n_2, we can obtain the following two first order conditions:

$$n_1 = \frac{2(\alpha_1 n_2 + \alpha_2 n_2 - f_1)}{t_1}, \tag{A11}$$

$$n_2 = \frac{2(\alpha_1 n_1 + \alpha_2 n_1 - f_2)}{t_2}. \tag{A12}$$

By solving the two equations above simultaneously, we obtain the socially optimal market shares:

$$n_1^* = \frac{2(2f_2(\alpha_1 + \alpha_2) + f_1 t_2)}{4(\alpha_1 + \alpha_2)^2 - t_1 t_2}, \tag{A13}$$

$$n_2^* = \frac{2(2f_1(\alpha_1 + \alpha_2) + f_2 t_1)}{4(\alpha_1 + \alpha_2)^2 - t_1 t_2}, \tag{A14}$$

Then, we plug n_1^*, n_2^* into W^S and obtain the following socially optimal welfare:

$$W^{S*} = -\frac{4 f_1 f_2(\alpha_1+\alpha_2)+ f_1^2 t_2 + f_2^2 t_1}{4(\alpha_1+\alpha_2)^2 - t_1 t_2}.$$

Since $n_1^*, n_2^* \in (0,1)$ requires $4(\alpha_1+\alpha_2)^2 - t_1 t_2 > 0$ by (A13) or (A14), we have $W^{S*} < 0$. Thus, the above result is not optimal for the social planner, since it is worse than $W^S = 0$ where the platform does not attract any agents on the two-sided market.

QED.

IV. Proof of Proposition 2

Step 1: We show that the socially optimal market outcome can only be in one of the following 4 cases: Case 1: $n_1^* = 0, n_2^* = 0$; Case 2: $n_1^* = 1, n_2^* = 1$; Case 3: $n_1^* = 1, n_2^* \in (0,1)$; and Case 4: $n_1^* \in (0,1), n_2^* = 1$.

It suffices to show if $n_1^* n_2^* = 0$, then $n_1^* = n_2^* = 0$. Since the social welfare is $(\alpha_1+\alpha_2)n_1 n_2 - (n_1 f_1 + n_2 f_2) - \frac{1}{4}(n_1^2 t_1 + n_2^2 t_2)$, when $n_1^* n_2^* = 0$, the welfare reduces to $-(n_1 f_1 + n_2 f_2) - \frac{1}{4}(n_1^2 t_1 + n_2^2 t_2)$, which is negative unless $n_1^* = n_2^* = 0$.

Step 2: We completely characterize the socially optimal outcome in each case.

Case 1: $n_1^* = 0, n_2^* = 0$

$n_1^* = 0, n_2^* = 0$ implies $n_1' = \frac{2(\alpha_1 n_2^* + \alpha_2 n_2^* - f_1)}{t_1} = -\frac{f_1}{t_1} \le 0$ and $n_2' = \frac{2(\alpha_1 n_1^* + \alpha_2 n_1^* - f_2)}{t_2} = -\frac{f_2}{t_2} \le 0$.

Both inequalities always hold since $n_1^* = 0, n_2^* = 0$. The socially optimal welfare is $W^{S_1 *} = 0$.

Case 2: $n_1^* = 1, n_2^* = 1$

$n_1^* = 1, n_2^* = 1$ implies $n_1' = \frac{2(\alpha_1 n_2^* + \alpha_2 n_2^* - f_1)}{t_1} = \frac{2(\alpha_1 + \alpha_2 - f_1)}{t_1} \ge 1$ and

$n_2' = \dfrac{2(\alpha_1 n_1^* + \alpha_2 n_1^* - f_2)}{t_2} = \dfrac{2(\alpha_1 + \alpha_2 - f_2)}{t_2} \geq 1$. The socially optimal welfare is $W^{S_2{}^*} = \alpha_1 + \alpha_2 - (f_1 + f_2) - \dfrac{1}{4}(t_1 + t_2)$. Note that $W^{S_2*} \geq 0$ requires $\alpha_1 + \alpha_2 \geq \dfrac{t_1 + t_2}{4} + f_1 + f_2$. Also, from the analyses below in Case 3 and Case 4, we will see that $n_1^* = 1$ requires $\alpha_1 + \alpha_2 \geq \dfrac{t_1}{2} + f_1$ and $n_2^* = 1$ requires $\alpha_1 + \alpha_2 \geq \dfrac{t_2}{2} + f_2$.

Case 3: $n_1^* = 1, n_2^* \in (0,1)$

$n_1^* = 1, n_2^* \in (0,1)$ implies $n_1' = \dfrac{2(\alpha_1 n_2^* + \alpha_2 n_2^* - f_1)}{t_1} \geq 1$ and $n_2^* = \dfrac{2(\alpha_1 n_1^* + \alpha_2 n_1^* - f_2)}{t_2}$ by first order conditions. Thus, $n_2^* = \dfrac{2(\alpha_1 + \alpha_2 - f_2)}{t_2}$, $W^{S_3*} = \dfrac{1}{4t_2}[4(\alpha_1 + \alpha_2 - f_2)^2 - (4 f_1 + t_1)t_2]$. Note that $n_2^* \in (0,1)$ requires $f_2 < \alpha_1 + \alpha_2 < \dfrac{t_2}{2} + f_2$. Also note that $W^{S_3*} > 0$ requires $\alpha_1 + \alpha_2 > f_2 + \dfrac{1}{2}\sqrt{(4f_1 + t_1)t_2}$.

Case 4: $n_2^* = 1, n_1^* \in (0,1)$

$n_2^* = 1, n_1^* \in (0,1)$ implies $n_2' = \dfrac{2(\alpha_1 n_1^* + \alpha_2 n_1^* - f_2)}{t_2} \geq 1$ and $n_1^* = \dfrac{2(\alpha_1 n_2^* + \alpha_2 n_2^* - f_1)}{t_1}$ by first order conditions. Thus, $n_1^* = \dfrac{2(\alpha_1 + \alpha_2 - f_1)}{t_1}$, $W^{S_4*} = \dfrac{1}{4t_1}[4(\alpha_1 + \alpha_2 - f_1)^2 - (4 f_2 + t_2)t_1]$. Note that $n_1^* \in (0,1)$ requires $f_1 < \alpha_1 + \alpha_2 < \dfrac{t_1}{2} + f_1$. Also note that $W^{S_4*} > 0$ requires $\alpha_1 + \alpha_2 > f_1 + \dfrac{1}{2}\sqrt{(4f_2 + t_2)t_1}$.

Also, by simple algebraic operations, we can show $W^{S_3{}^*} - W^{S_2{}^*} = \dfrac{(\alpha_1 + \alpha_2 - f_2 - \frac{1}{2}t_2)^2}{t_2} \geq 0$

and $W^{S_4{}^*} - W^{S_2{}^*} = \dfrac{(\alpha_1 + \alpha_2 - f_1 - \frac{1}{2}t_1)^2}{t_1} \geq 0$, where the two equalities hold only when

$\alpha_1 + \alpha_2 = \dfrac{t_2}{2} + f_2$ and $\alpha_1 + \alpha_2 = \dfrac{t_1}{2} + f_1$, respectively.

Step 3: We completely characterize the socially optimal outcome under full parameter space.

SI) $t_2 \geq 4f_1 + t_1$:

In this scenario, we have $\sqrt{(4f_1 + t_1)t_2} \leq t_2$ and $\dfrac{1}{2}t_2 + f_2 \geq \dfrac{1}{2}t_1 + f_1$.

Note that Case 4 is not optimal since $\alpha_1 + \alpha_2 < \dfrac{t_1}{2} + f_1$ and $\alpha_1 + \alpha_2 > f_1 + \dfrac{1}{2}\sqrt{(4f_2 + t_2)t_1}$ imply $4f_2 + t_2 < t_1$, which contradicts with $t_2 \geq 4f_1 + t_1$.

SI-i) $\alpha_1 + \alpha_2 < f_2 + \dfrac{1}{2}\sqrt{(4f_1 + t_1)t_2}$:

Case 3 is not optimal since $\alpha_1 + \alpha_2 < f_2 + \dfrac{1}{2}\sqrt{(4f_1 + t_1)t_2}$ implies $W^{S_3*} < 0$. Case 2 is not optimal since $W^{S_2*} \leq W^{S_3*} < 0$. Therefore, the optimal solution is characterized by Case 1.

SI-ii) $f_2 + \dfrac{1}{2}\sqrt{(4f_1 + t_1)t_2} \leq \alpha_1 + \alpha_2 < \dfrac{1}{2}t_2 + f_2$:

Case 2 is not optimal since $\alpha_1 + \alpha_2 < \dfrac{1}{2}t_2 + f_2$ implies $n_2^* < 1$. Case 1 is not optimal since $\alpha_1 + \alpha_2 \geq f_2 + \dfrac{1}{2}\sqrt{(4f_1 + t_1)t_2}$ implies $W^{S_3*} \geq 0$. Therefore, the optimal solution is characterized by Case 3.

SI-iii) $\alpha_1 + \alpha_2 \geq \dfrac{1}{2}t_2 + f_2$:

Case 3 is not optimal since $n_2^* \in (0,1)$ requires $\alpha_1 + \alpha_2 < \dfrac{1}{2}t_2 + f_2$, contradicting

$\alpha_1+\alpha_2 \geq \frac{1}{2}t_2 + f_2$. Case 1 is not optimal since $\alpha_1+\alpha_2 \geq \frac{1}{2}t_2 + f_2$ implies $\alpha_1+\alpha_2 \geq \frac{t_1+t_2}{4}+f_1+f_2$ given $t_2 \geq 4f_1+t_1$, and $\alpha_1+\alpha_2 \geq \frac{t_1+t_2}{4}+f_1+f_2$ implies $W^{S_2*} \geq 0$. Therefore, the optimal solution is characterized by Case 2.

SII) $2f_1-2f_2+t_1 \leq t_2 < 4f_1+t_1$:

In this scenario, we have $\sqrt{(4f_1+t_1)t_2} > t_2$ and $\frac{1}{2}t_2 + f_2 \geq \frac{1}{2}t_1 + f_1$.

Note that Case 4 is not optimal since $\alpha_1+\alpha_2 < \frac{1}{2}t_1 + f_1$ and $\alpha_1+\alpha_2 > f_1+\frac{1}{2}\sqrt{(4f_2+t_2)t_1}$ imply $4f_2+t_2 < t_1$, which contradicts with $2f_1-2f_2+t_1 \leq t_2$.

Also note that Case 3 is not optimal since $\alpha_1+\alpha_2 < \frac{1}{2}t_2 + f_2$ and $\alpha_1+\alpha_2 > f_2+\frac{1}{2}\sqrt{(4f_1+t_1)t_2}$ imply $t_2 > 4f_1+t_1$, which contradicts with $t_2 < 4f_1+t_1$.

SII-i) $\alpha_1+\alpha_2 < \frac{t_1+t_2}{4}+f_1+f_2$:

Case 2 is not optimal since $\alpha_1+\alpha_2 < \frac{t_1+t_2}{4}+f_1+f_2$ implies $W^{S_2*} < 0$. Therefore, the optimal solution is characterized by Case 1.

SII-ii) $\alpha_1+\alpha_2 \geq \frac{t_1+t_2}{4}+f_1+f_2$:

$\alpha_1+\alpha_2 \geq \frac{t_1+t_2}{4}+f_1+f_2$ and $t_2 < 4f_1+t_1$ imply that $\alpha_1+\alpha_2 \geq f_2+\frac{1}{2}t_2$. $\alpha_1+\alpha_2 \geq f_2+\frac{1}{2}t_2$ and $\frac{1}{2}t_2 + f_2 \geq \frac{1}{2}t_1 + f_1$ imply $\alpha_1+\alpha_2 \geq f_1+\frac{1}{2}t_1$. Thus we have $\pi^{M_2} \geq 0$ and $n_1^*=1, n_2^*=1$. Therefore, the optimal solution is characterized by Case 2.

Similarly, we can show the results when $\frac{1}{2}t_2 + f_2 < \frac{1}{2}t_1 + f_1$.

SIII) $-4f_2 + t_1 \le t_2 < 2f_1 - 2f_2 + t_1$

In this scenario, we have $\sqrt{(4f_2 + t_2)t_1} \ge t_1$ and $\frac{1}{2}t_2 + f_2 < \frac{1}{2}t_1 + f_1$.

Note that Case 3 is not optimal since $\alpha_1 + \alpha_2 < \frac{1}{2}t_2 + f_2$ and $\alpha_1 + \alpha_2 > f_2 + \frac{1}{2}\sqrt{(4f_1 + t_1)t_2}$ imply $t_2 > 4f_1 + t_1$, which contradicts with $t_2 < 2f_1 - 2f_2 + t_1$.

Also note that Case 4 is not optimal since $\alpha_1 + \alpha_2 < \frac{1}{2}t_1 + f_1$ and $\alpha_1 + \alpha_2 > f_1 + \frac{1}{2}\sqrt{(4f_2 + t_2)t_1}$ imply $4f_2 + t_2 < t_1$, which contradicts with $-4f_2 + t_1 \le t_2$.

SIII-i) $\alpha_1 + \alpha_2 < \frac{t_1+t_2}{4} + f_1 + f_2$:

Case 2 is not optimal since $\alpha_1 + \alpha_2 < \frac{t_1+t_2}{4} + f_1 + f_2$ implies $W^{S_2*} < 0$. Therefore, the optimal solution is characterized by Case 1.

SIII-ii) $\alpha_1 + \alpha_2 \ge \frac{t_1+t_2}{4} + f_1 + f_2$:

$\alpha_1 + \alpha_2 \ge \frac{t_1+t_2}{4} + f_1 + f_2$ and $-4f_2 + t_1 \le t_2$ imply that $\alpha_1 + \alpha_2 \ge f_1 + \frac{1}{2}t_1$. $\alpha_1 + \alpha_2 \ge f_1 + \frac{1}{2}t_1$ and $\frac{1}{2}t_2 + f_2 < \frac{1}{2}t_1 + f_1$ imply $\alpha_1 + \alpha_2 > f_2 + \frac{1}{2}t_2$. Thus we have $\pi^{M_2} \ge 0$ and $n_1^* = 1, n_2^* = 1$. Therefore, the optimal solution is characterized by Case 2.

SIV) $t_2 < -4f_2 + t_1$

In this scenario, we have $\sqrt{(4f_2+t_2)t_1} < t_1$ and $\frac{1}{2}t_2 + f_2 < \frac{1}{2}t_1 + f_1$.

Note that Case 3 is not optimal since $\alpha_1+\alpha_2 < \frac{1}{2}t_2 + f_2$ and $\alpha_1+\alpha_2 > f_2 + \frac{1}{2}\sqrt{(4f_1+t_1)t_2}$ imply $t_2 > 4f_1 + t_1$, which contradicts with $t_2 < -4f_2 + t_1$.

SIV-i) $\alpha_1+\alpha_2 < f_1 + \frac{1}{2}\sqrt{(4f_2+t_2)t_1}$:

Case 4 is not optimal since $\alpha_1+\alpha_2 < f_1 + \frac{1}{2}\sqrt{(4f_2+t_2)t_1}$ implies $W^{S_4*} < 0$. Case 2 is not optimal since $W^{S_2*} \le W^{S_4*} < 0$. Therefore, the optimal solution is characterized by Case 1.

SIV-ii) $f_1 + \frac{1}{2}\sqrt{(4f_2+t_2)t_1} \le \alpha_1+\alpha_2 < \frac{1}{2}t_1 + f_1$:

Case 2 is not optimal since $\alpha_1+\alpha_2 < \frac{1}{2}t_1 + f_1$ implies $n_1^* < 1$. Case 1 is not optimal since $\alpha_1+\alpha_2 \ge f_1 + \frac{1}{2}\sqrt{(4f_2+t_2)t_1}$ implies $W^{S_4*} \ge 0$. Therefore, the optimal solution is characterized by Case 4.

SIV-iii) $\alpha_1+\alpha_2 \ge \frac{1}{2}t_1 + f_1$:

Case 4 is not optimal since $n_1^* \in (0,1)$ requires $\alpha_1+\alpha_2 < \frac{1}{2}t_1 + f_1$, contradicting $\alpha_1+\alpha_2 \ge \frac{1}{2}t_1 + f_1$. Case 1 is not optimal since $\alpha_1+\alpha_2 \ge \frac{1}{2}t_1 + f_1$ implies $\alpha_1+\alpha_2 > \frac{t_1+t_2}{4} + f_1 + f_2$ given $t_2 < -4f_2 + t_1$, and $\alpha_1+\alpha_2 > \frac{t_1+t_2}{4} + f_1 + f_2$ implies $W^{S_2*} > 0$. Therefore, the optimal solution is characterized by Case 2.

QED.

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